Charles-Augustin COULOMB - A geotechnical tribute

Paris, september 25 & 26, 2023

Computational Limit Analysis and Extensions

K. Krabbenhoft

Optum Computational Engineering

Outline

- Introduction
- The conventional FE approach
- The Optum approach
- Examples
- Future developments

Introduction

Question:

What is the maximum magnitude of loading that the foundation can sustain?

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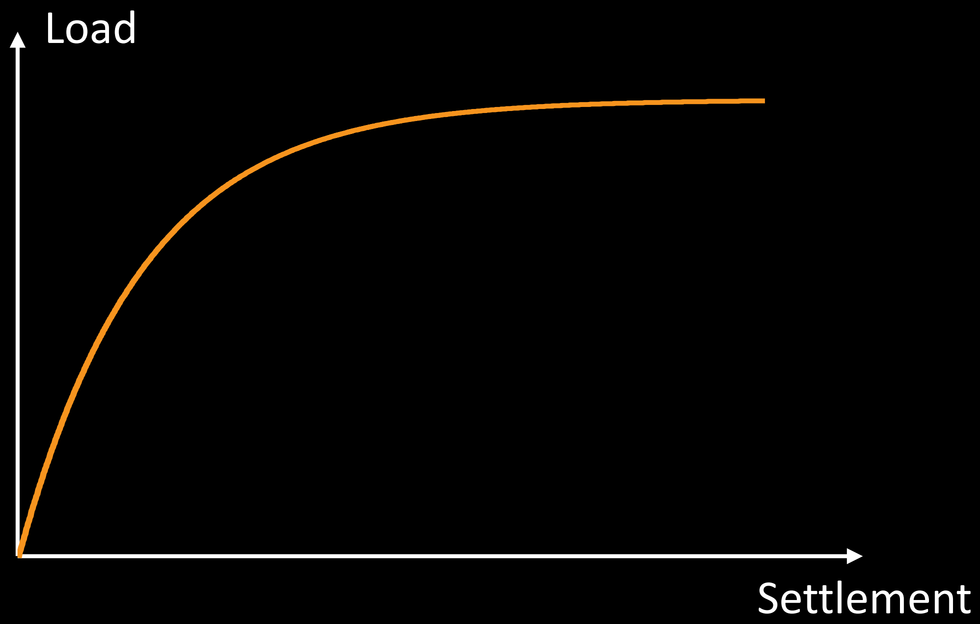

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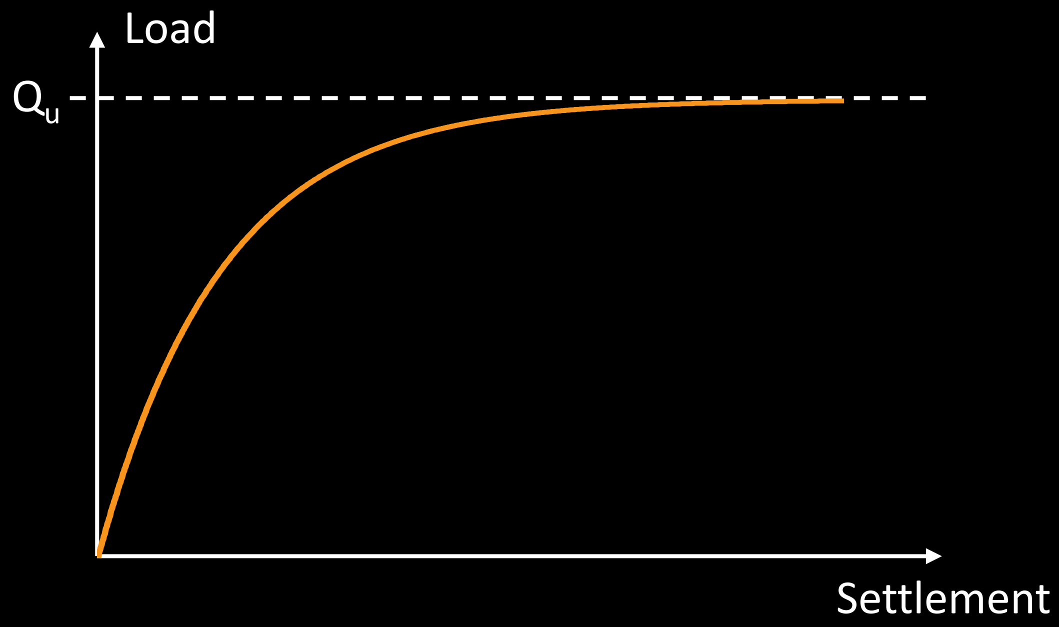

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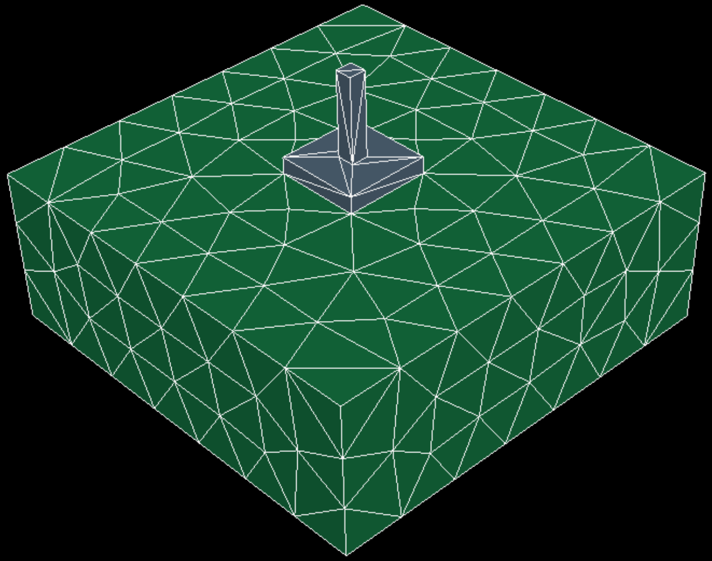

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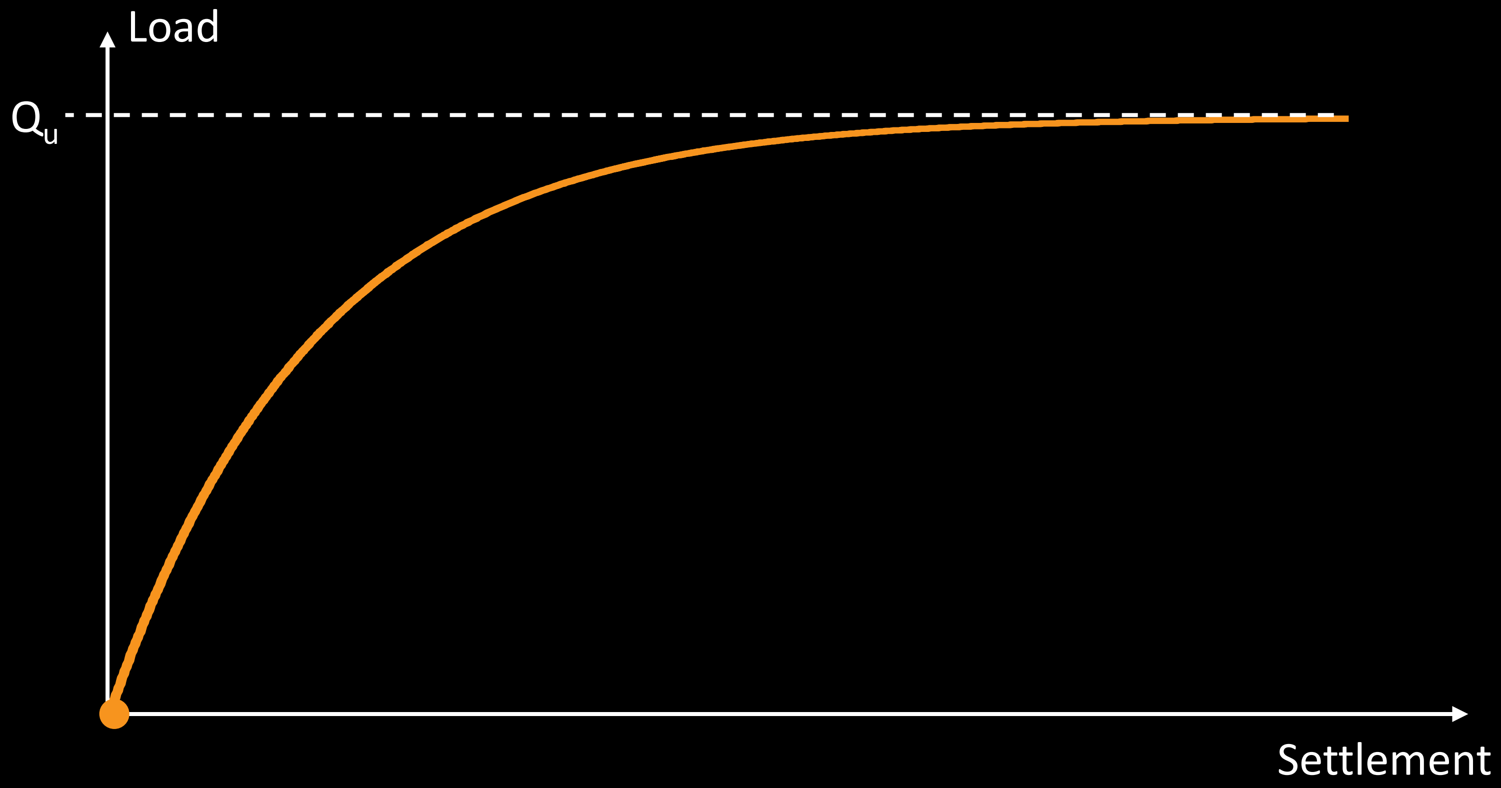

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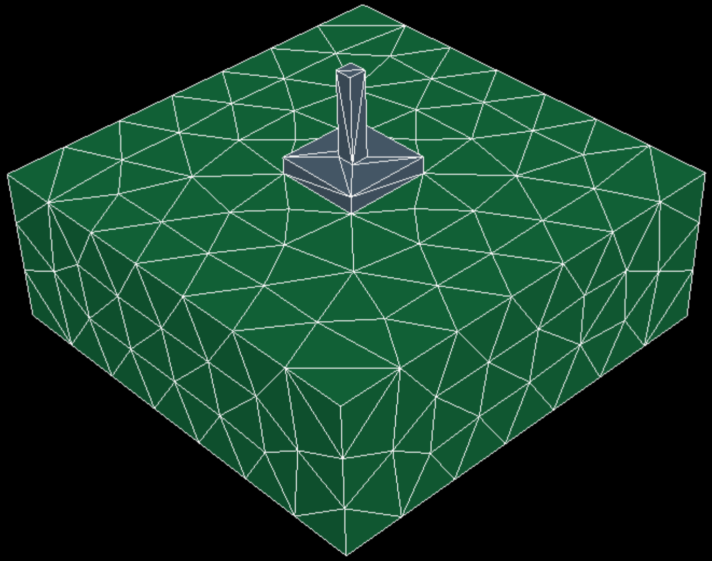

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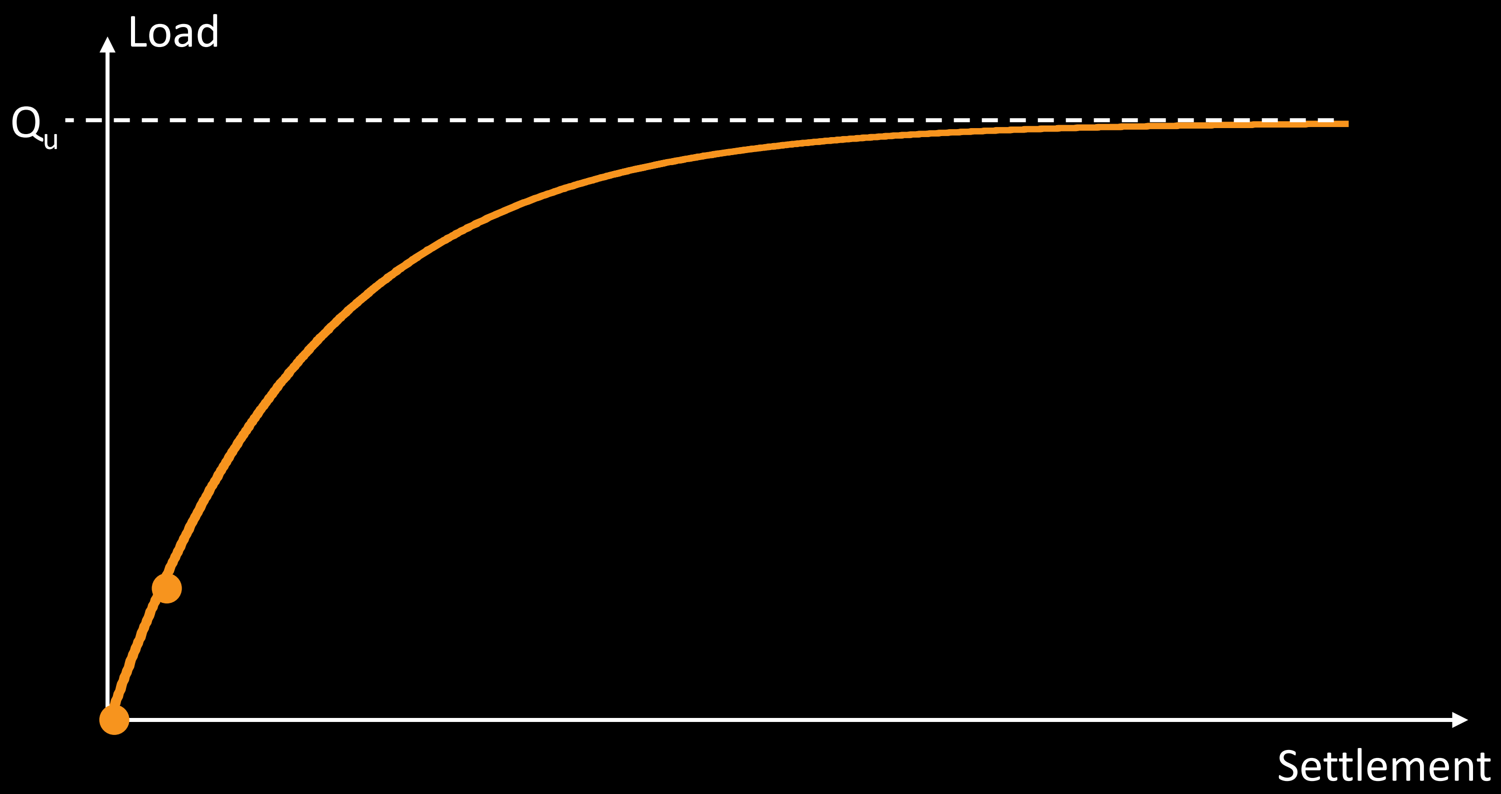

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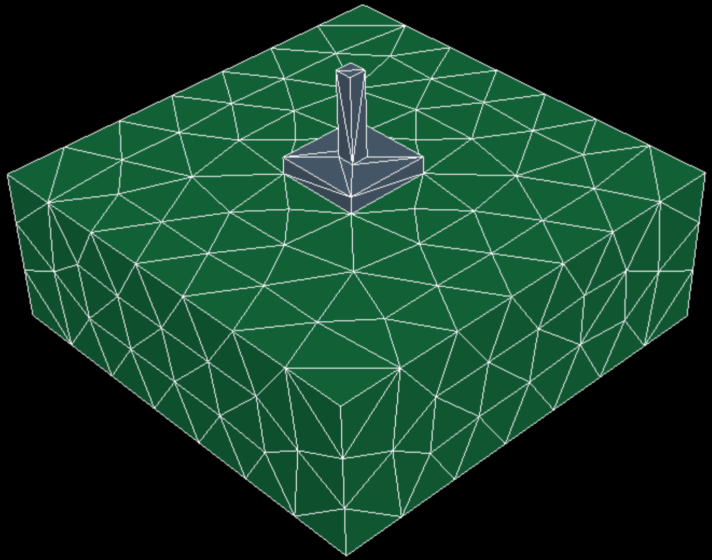

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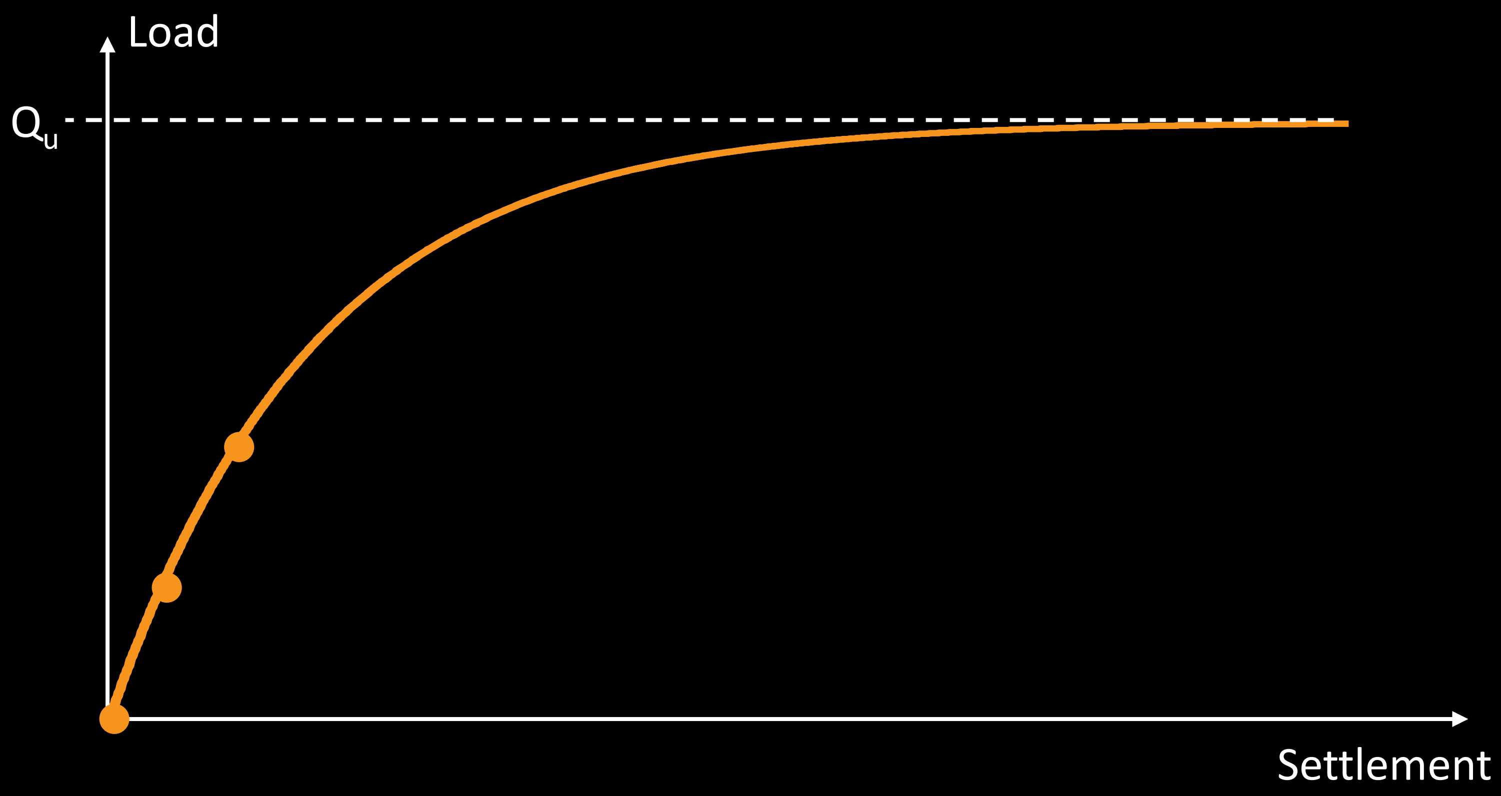

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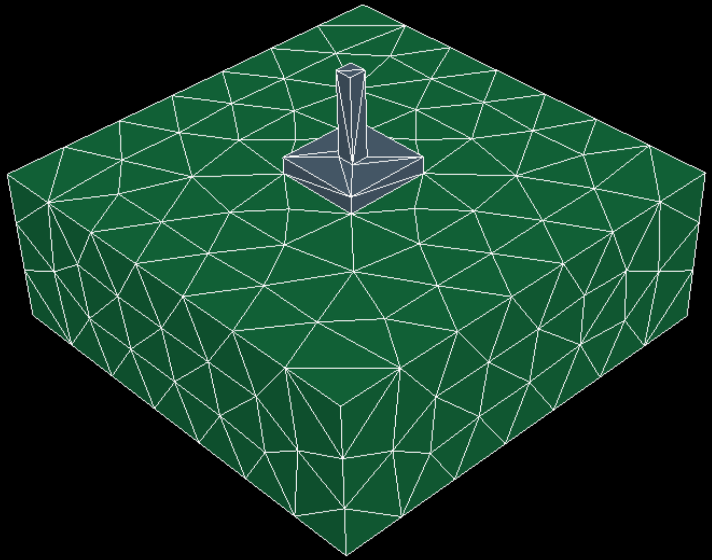

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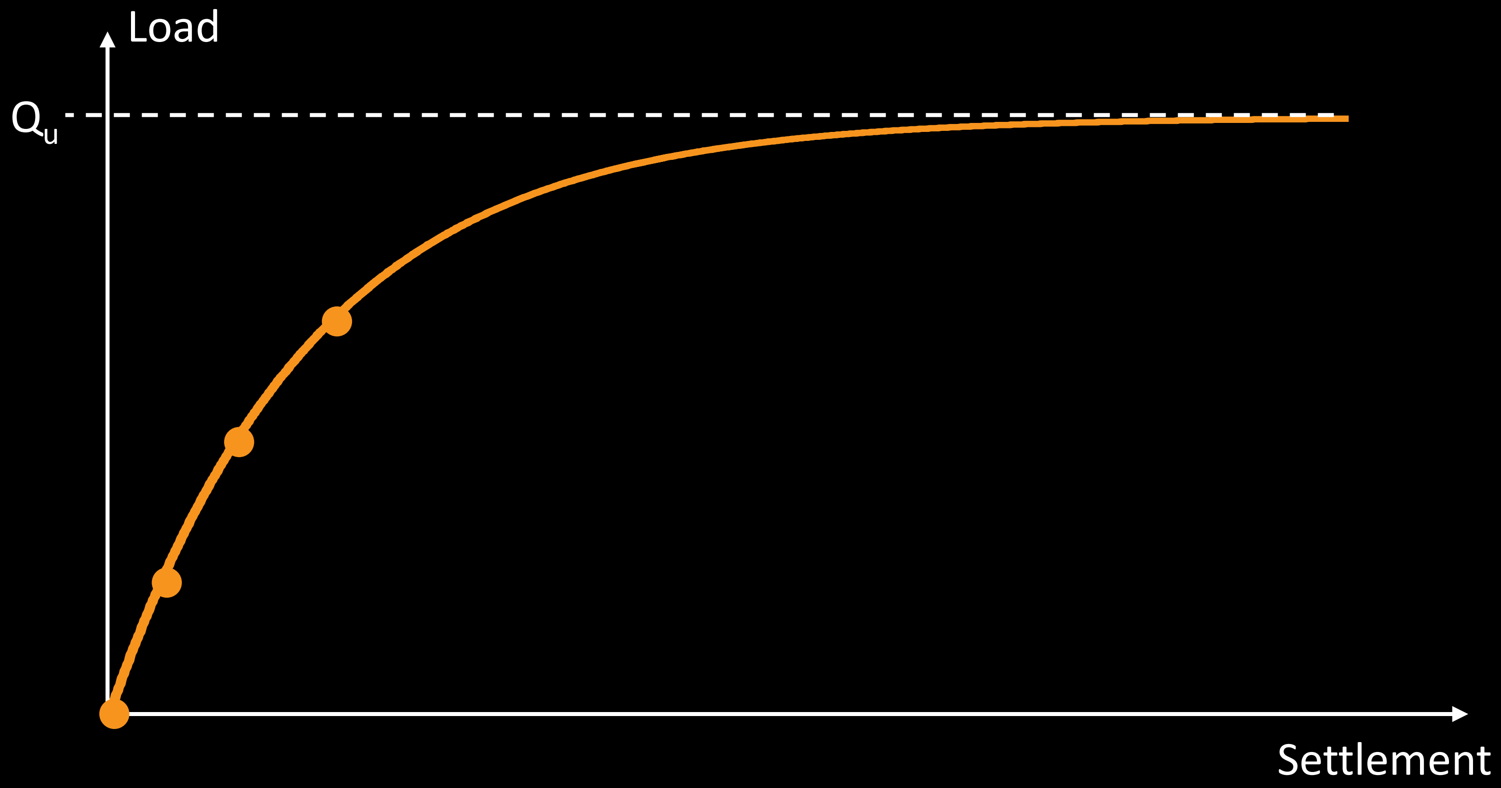

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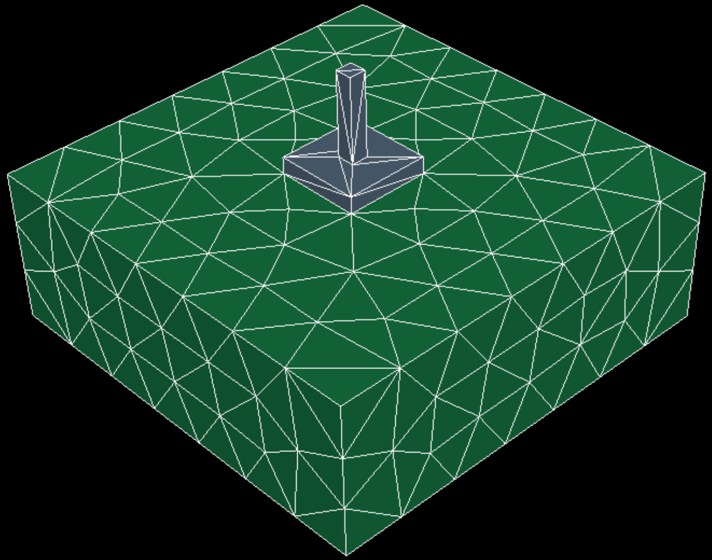

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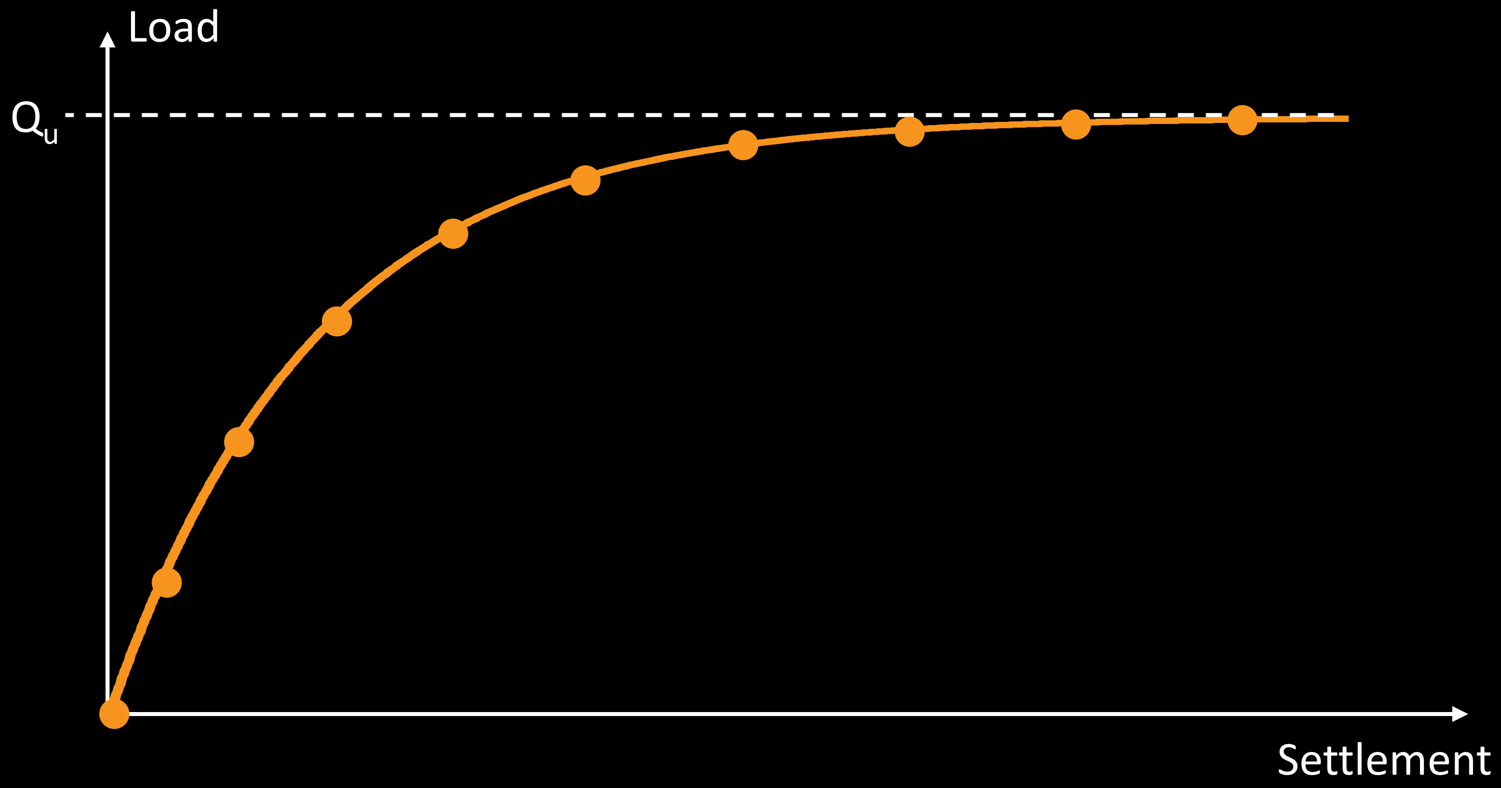

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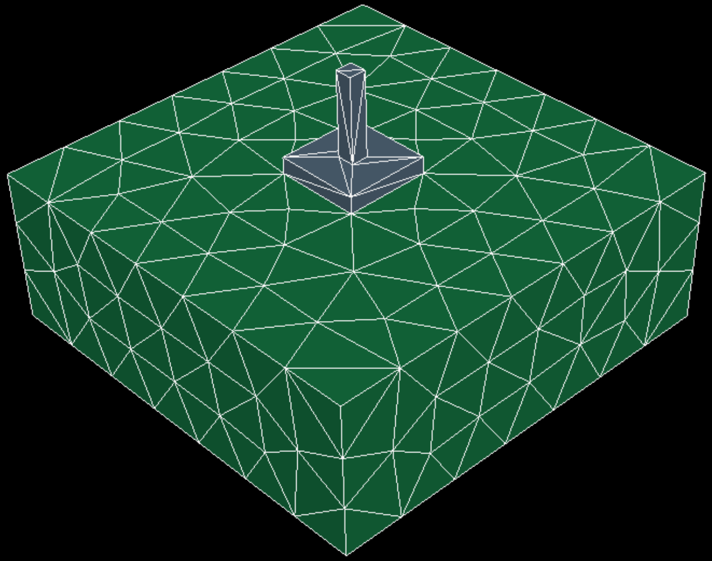

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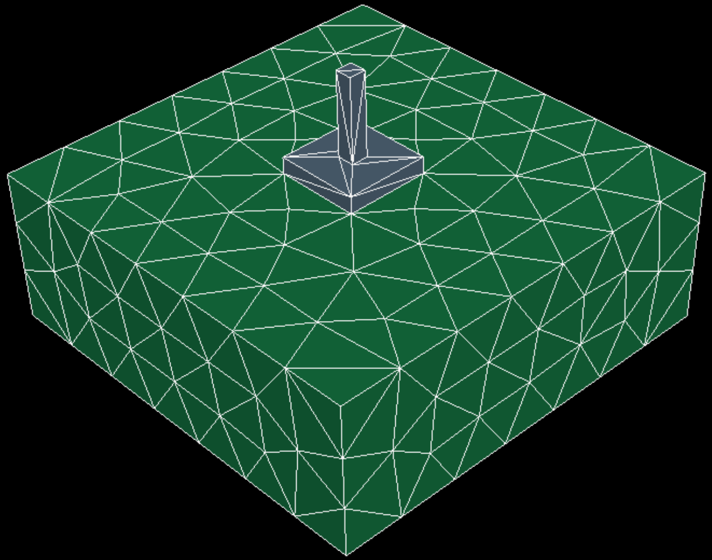

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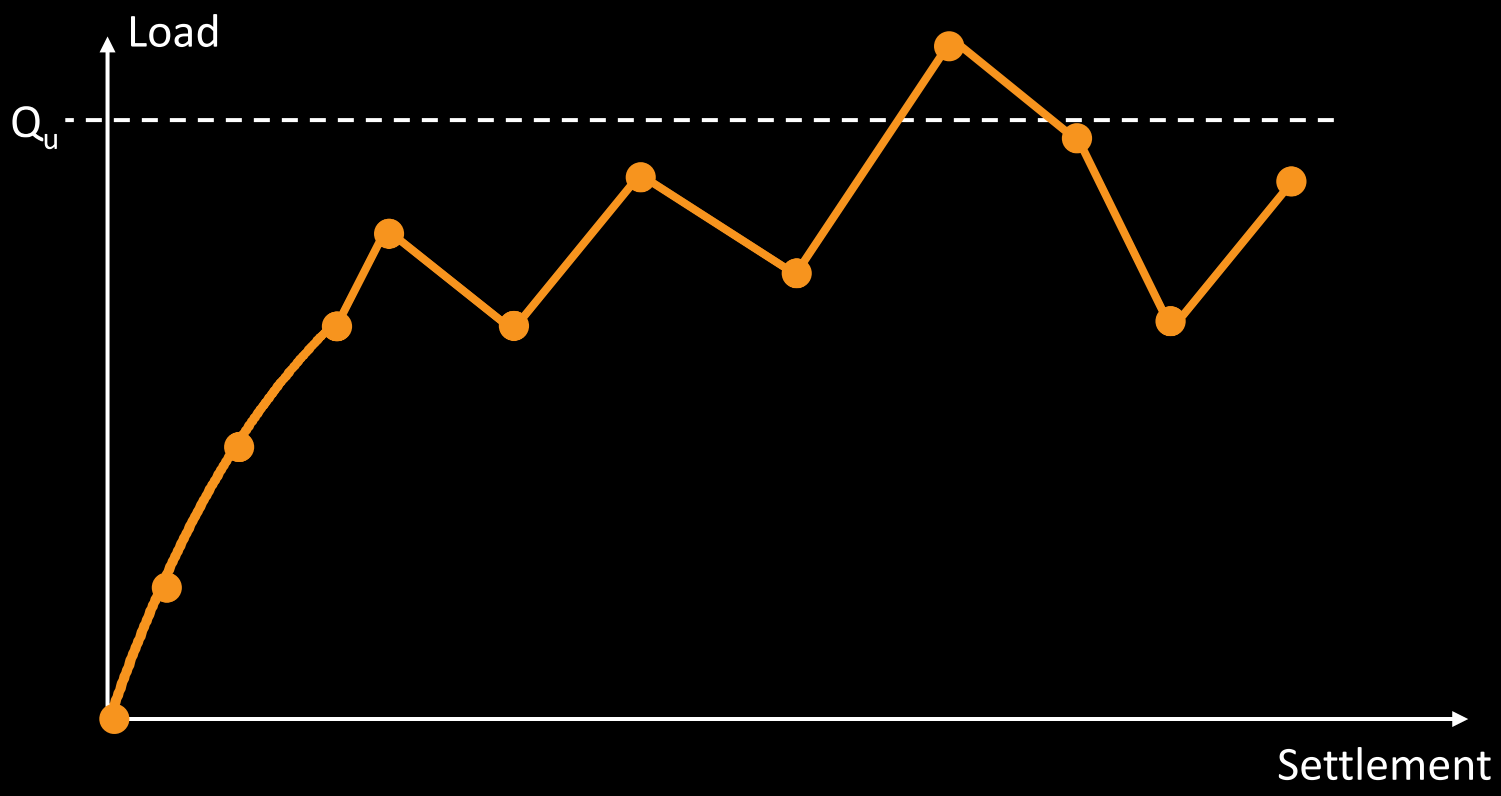

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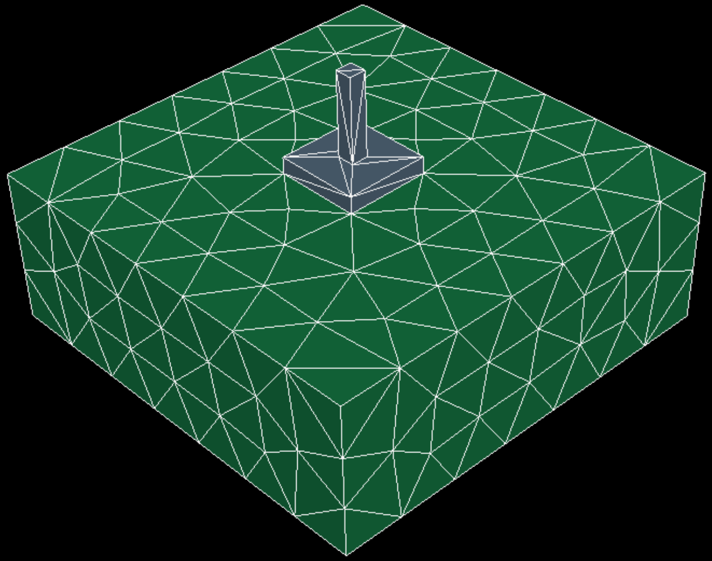

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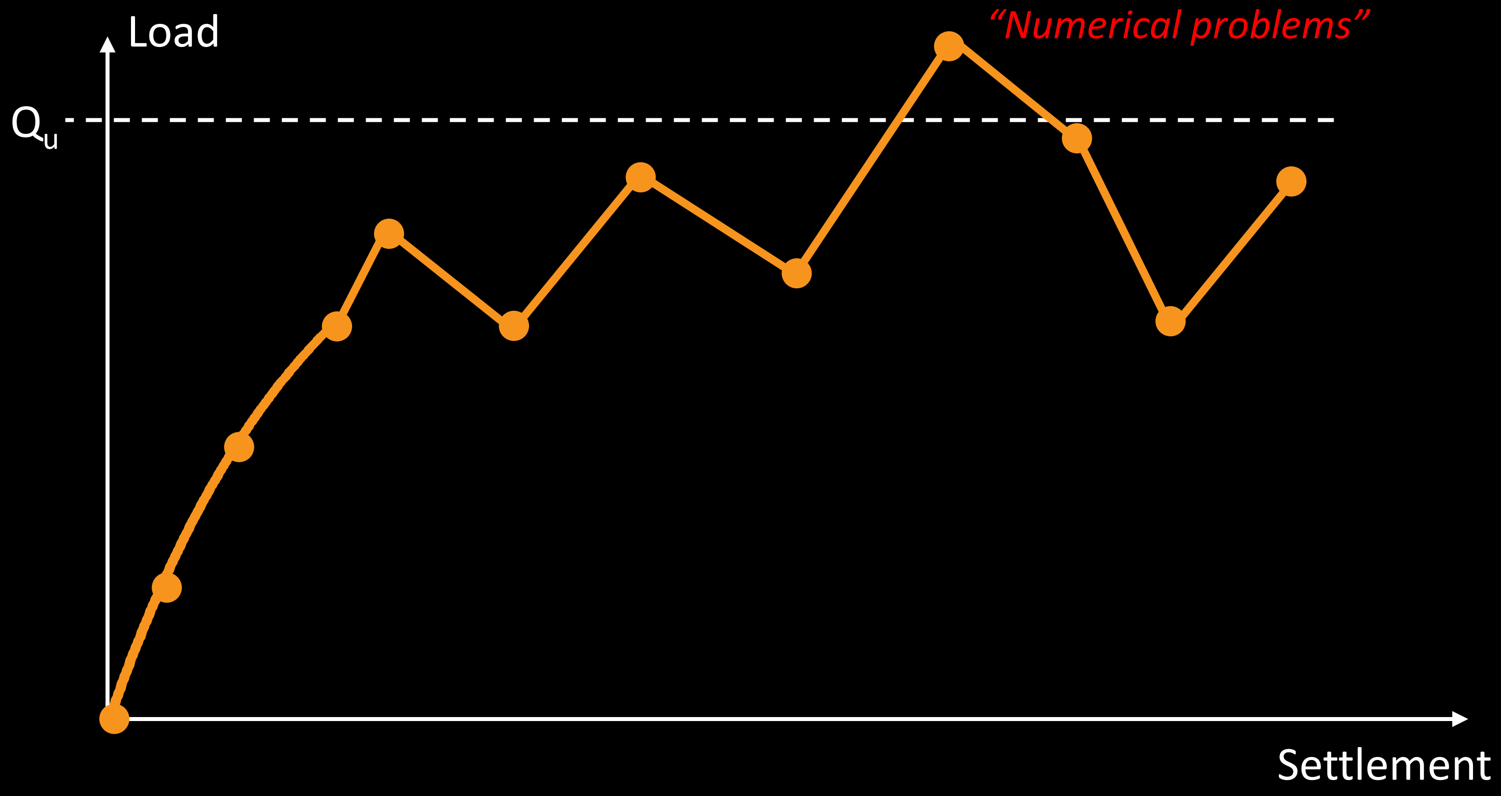
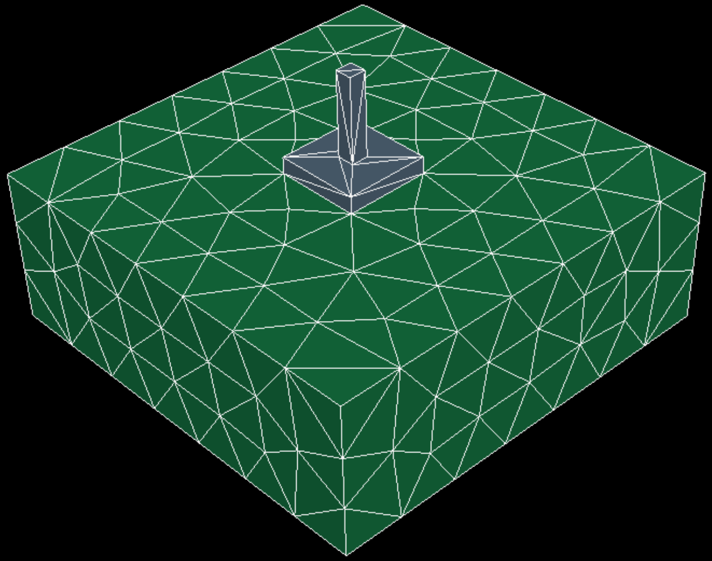


What is the limit load?

The conventional FE approach



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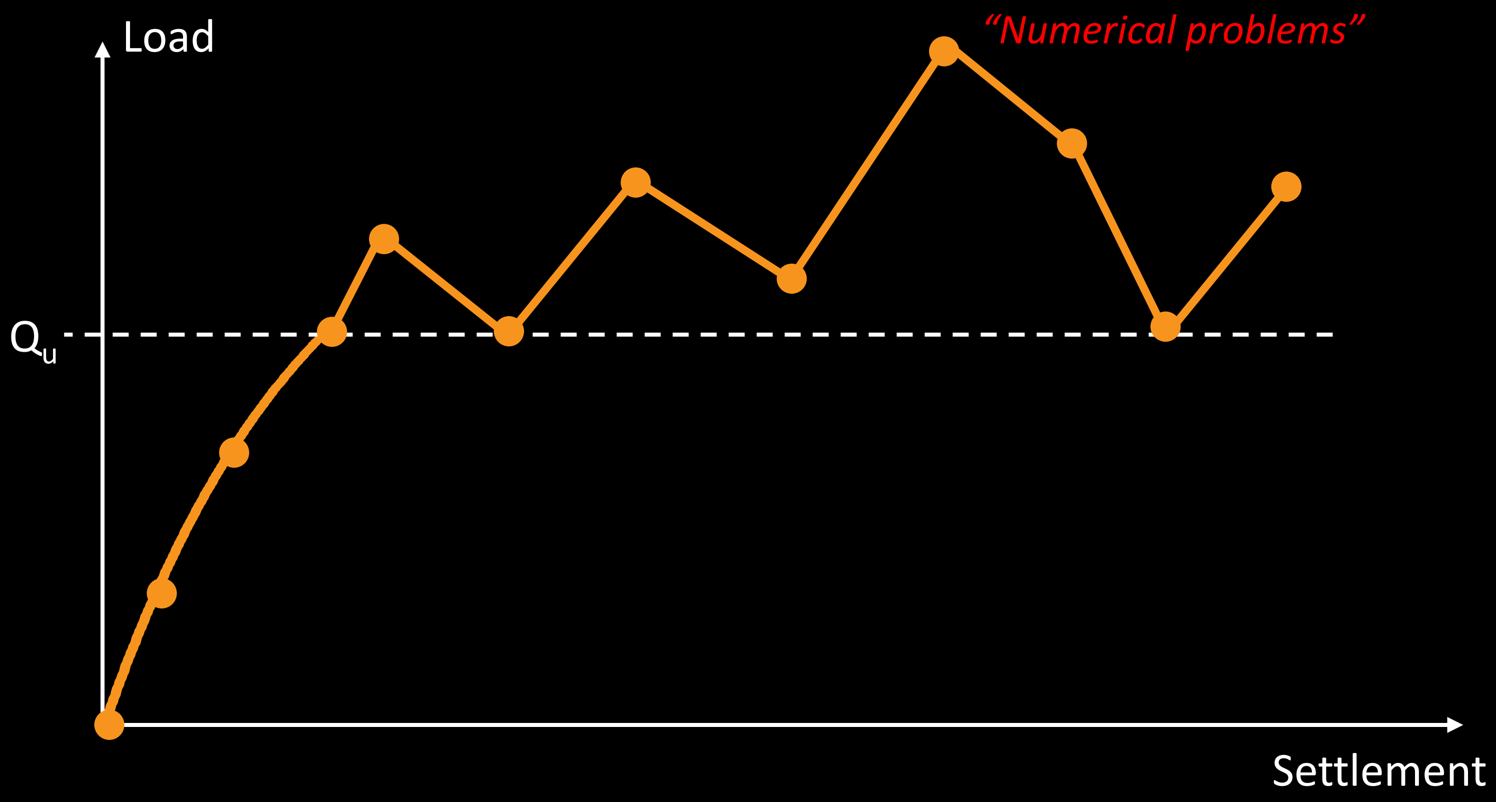

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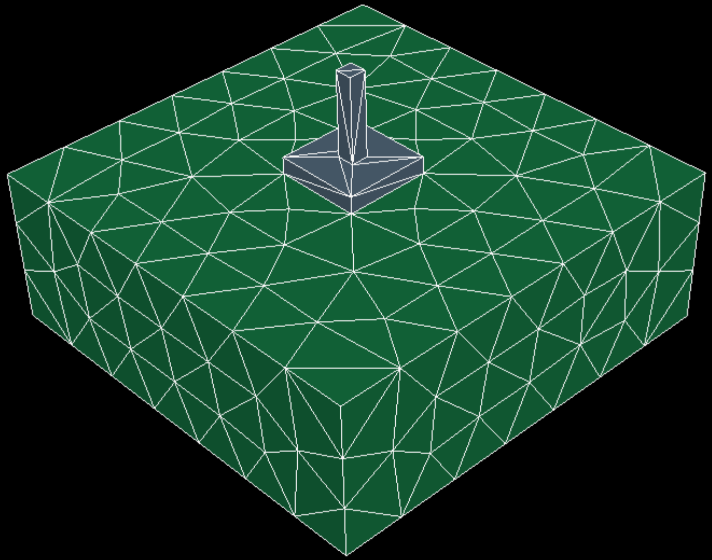

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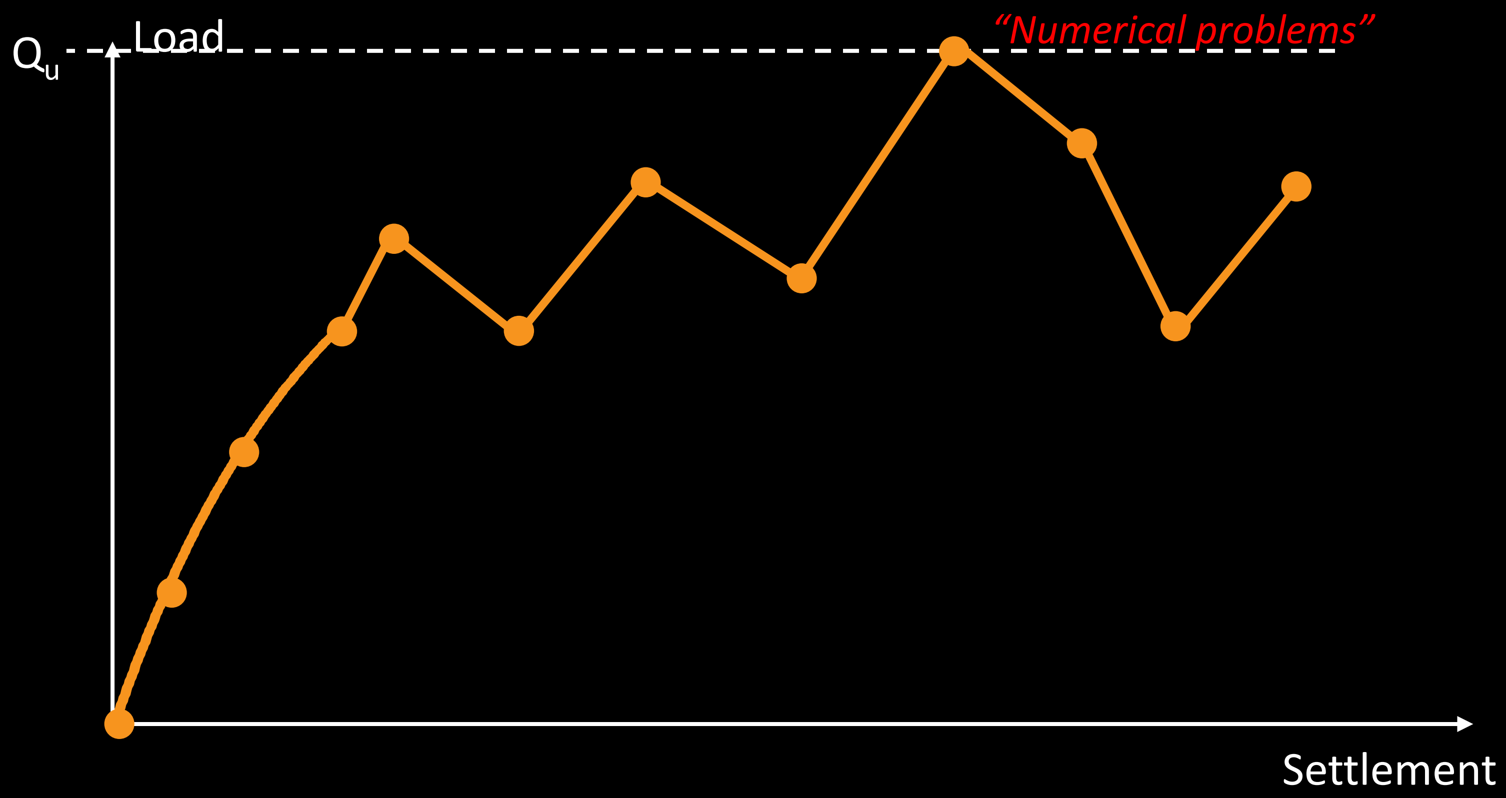

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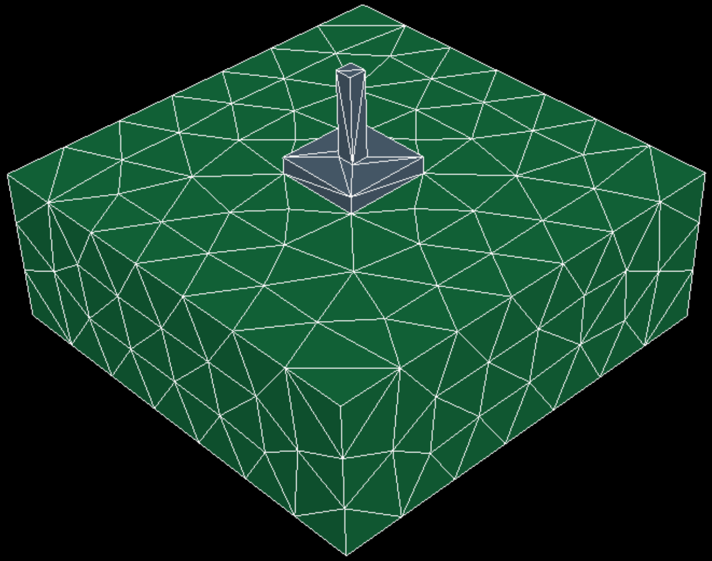

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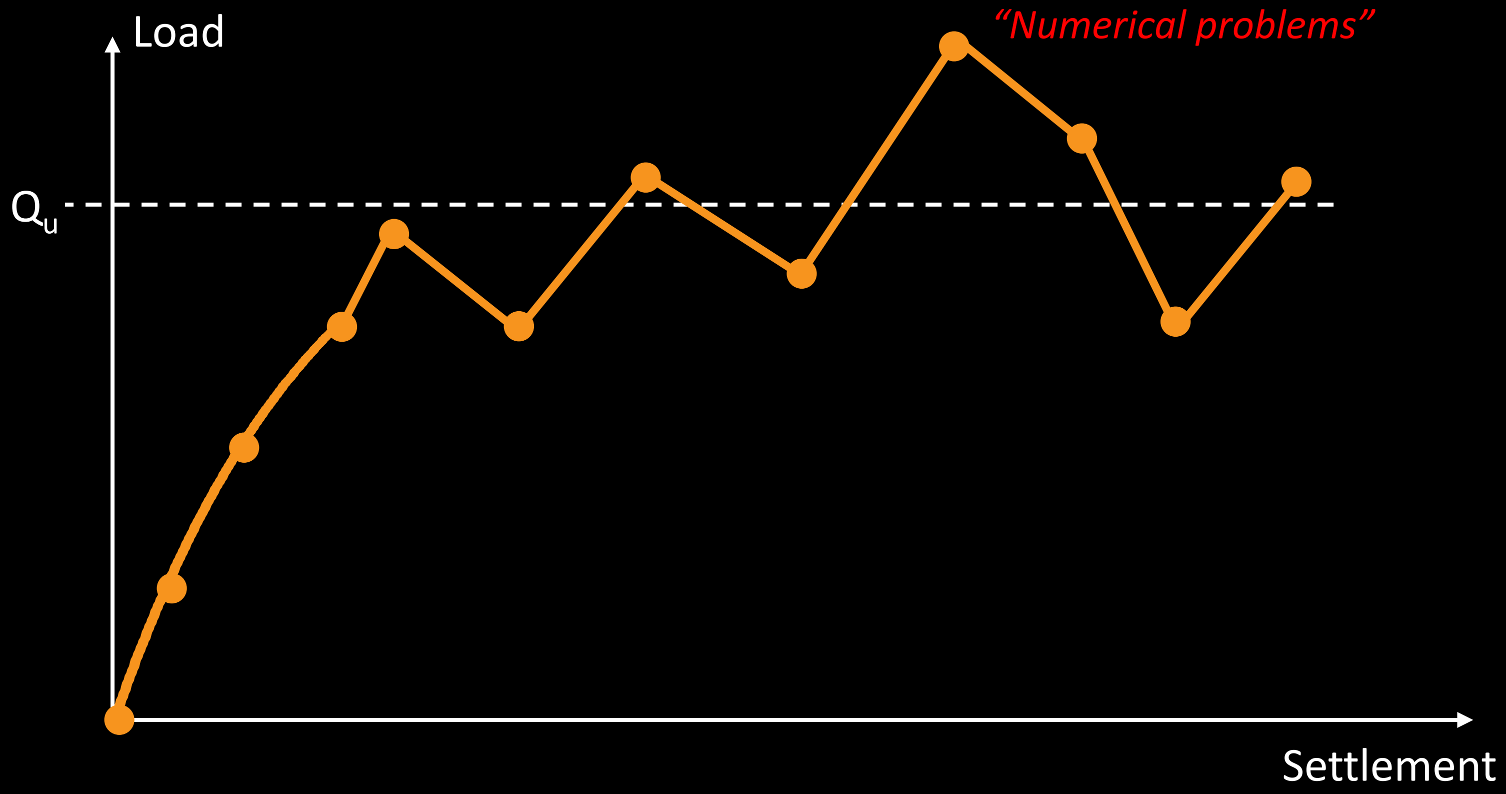

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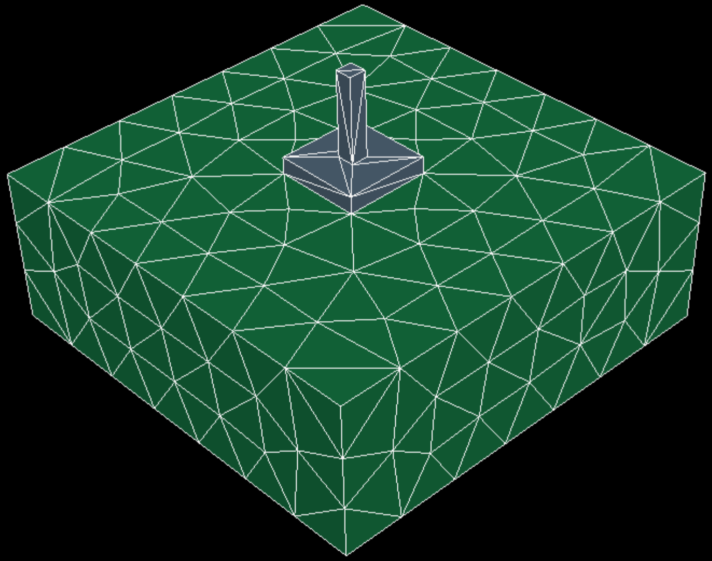

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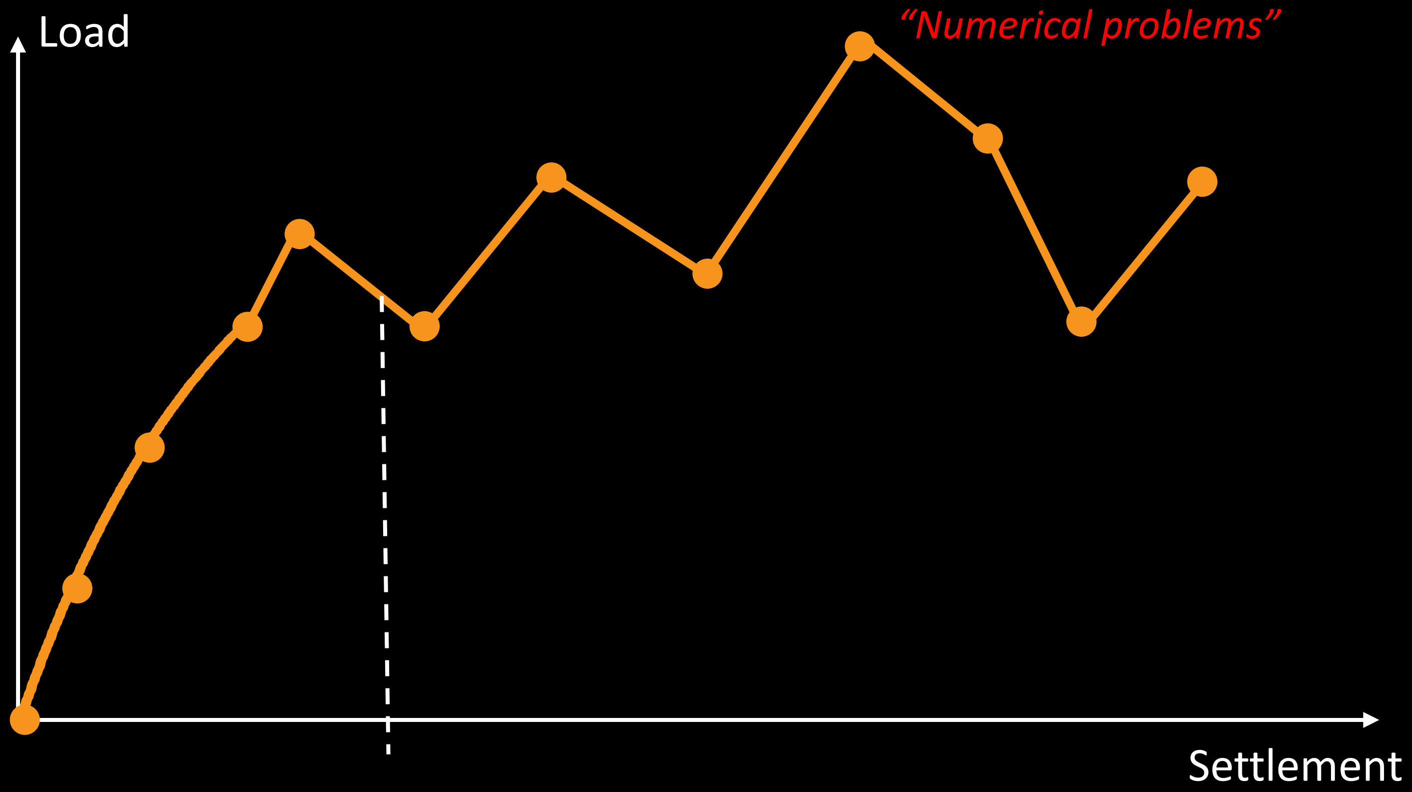

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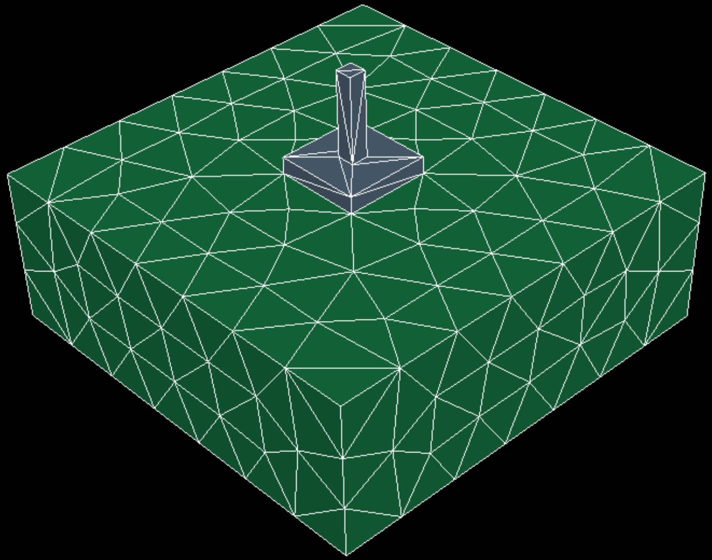

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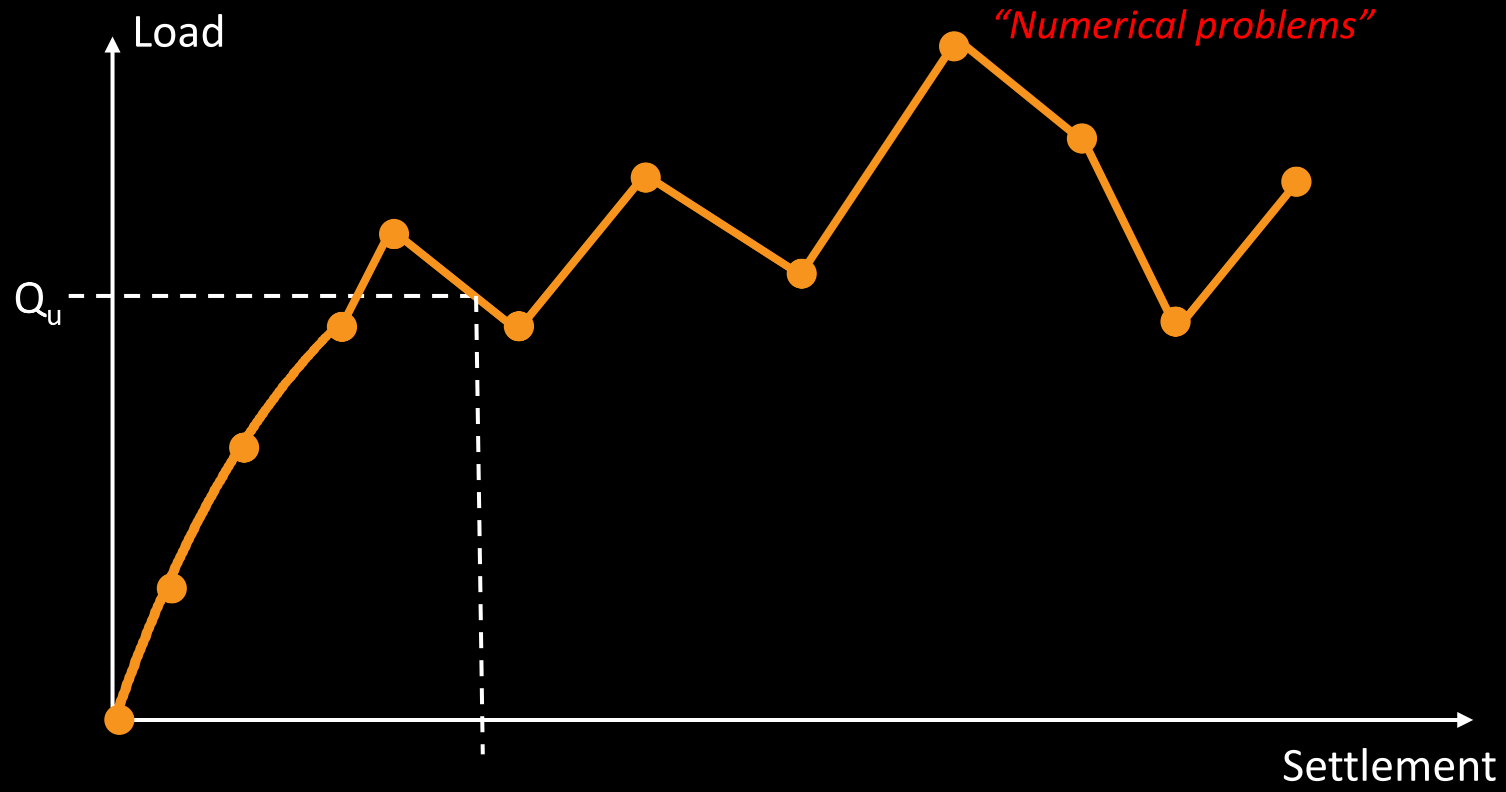

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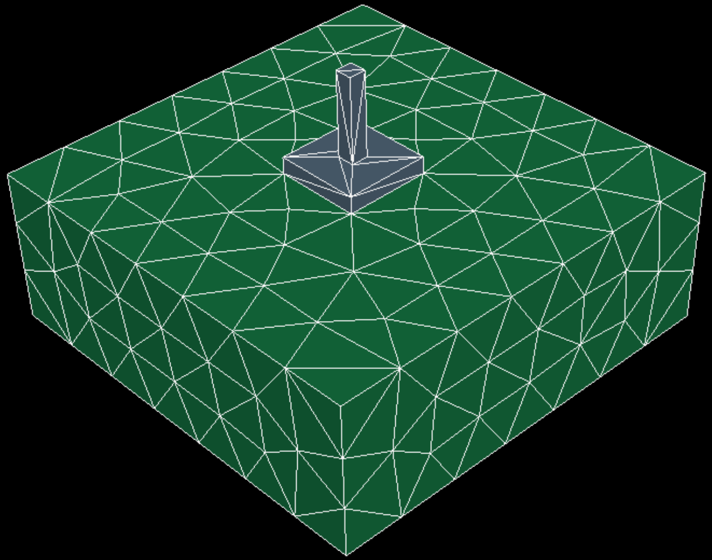

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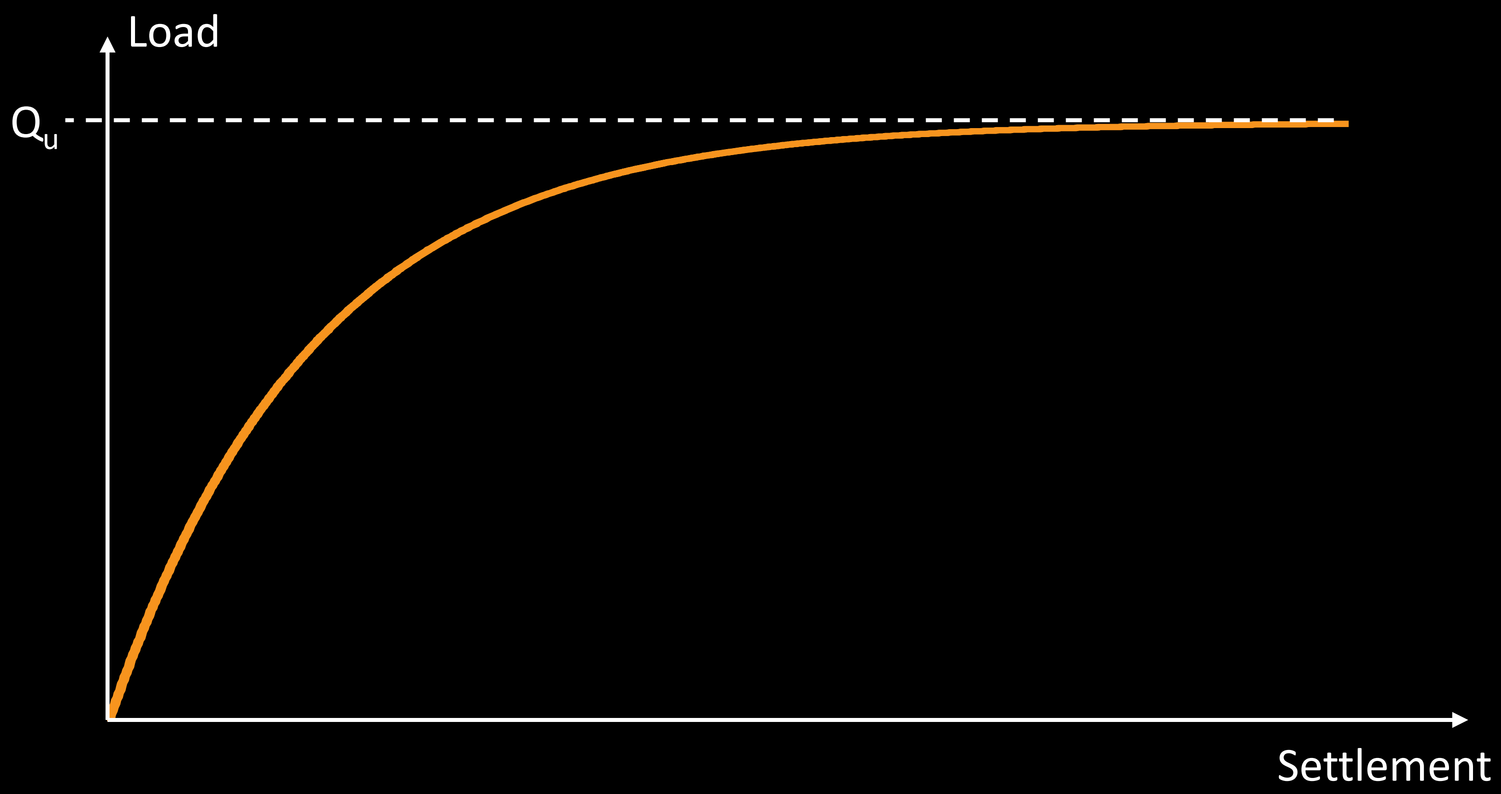

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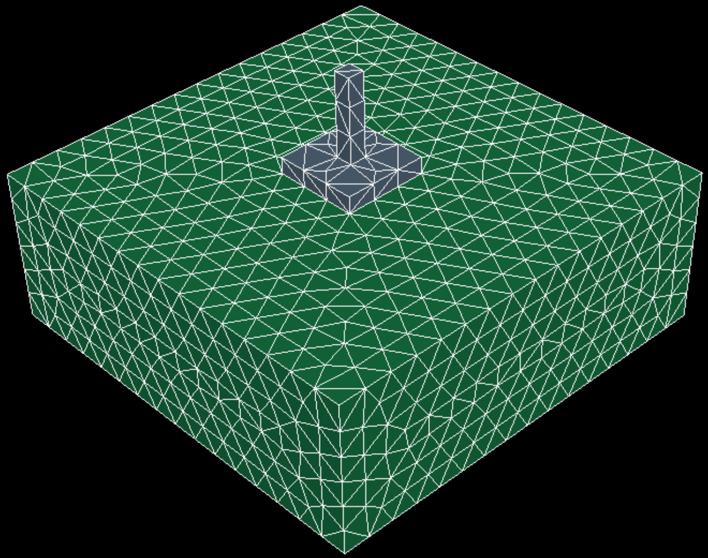

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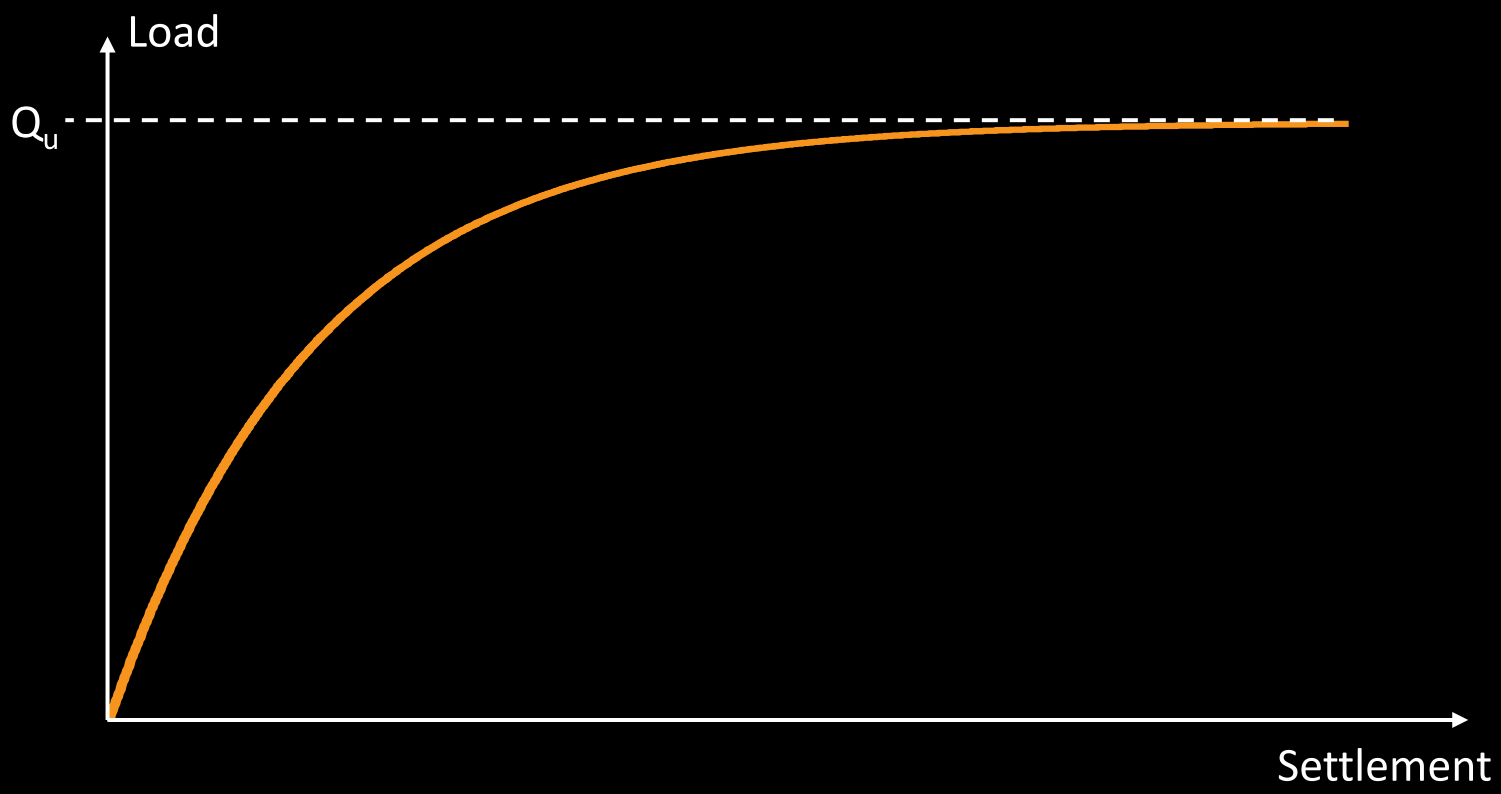

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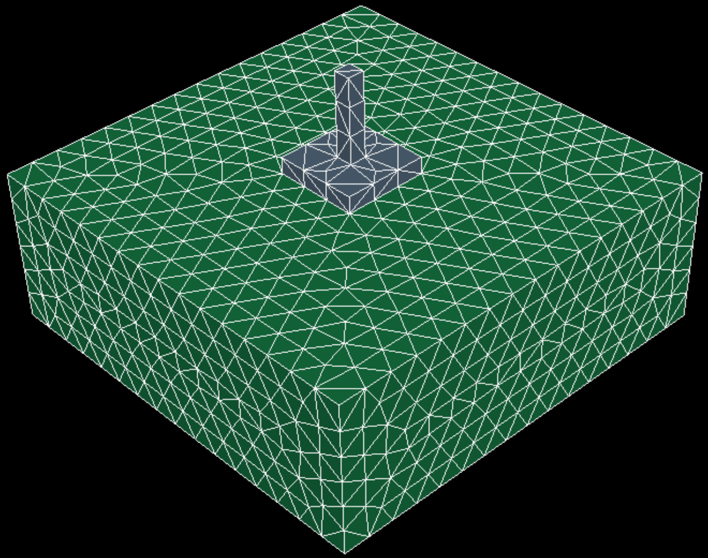

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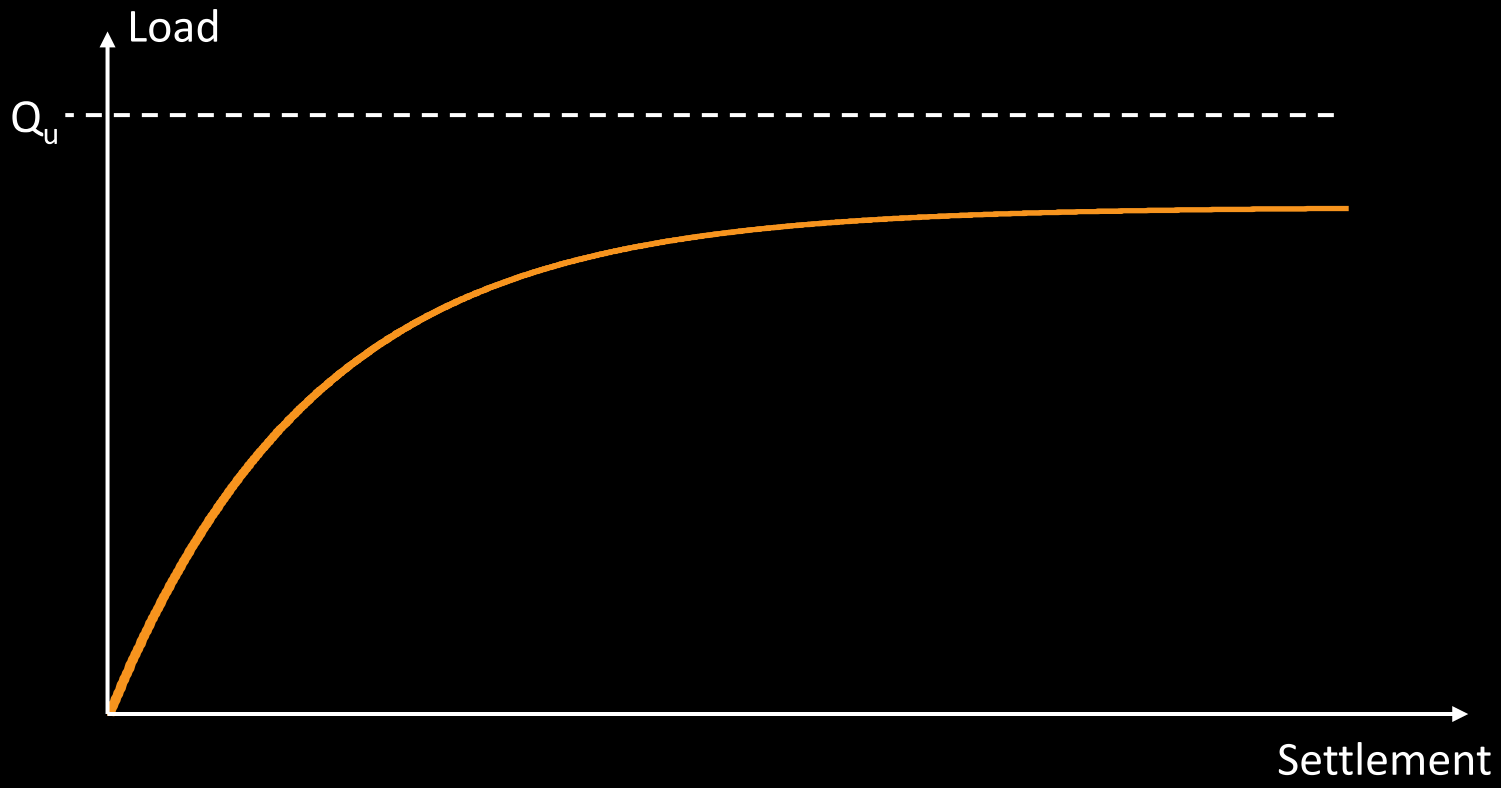

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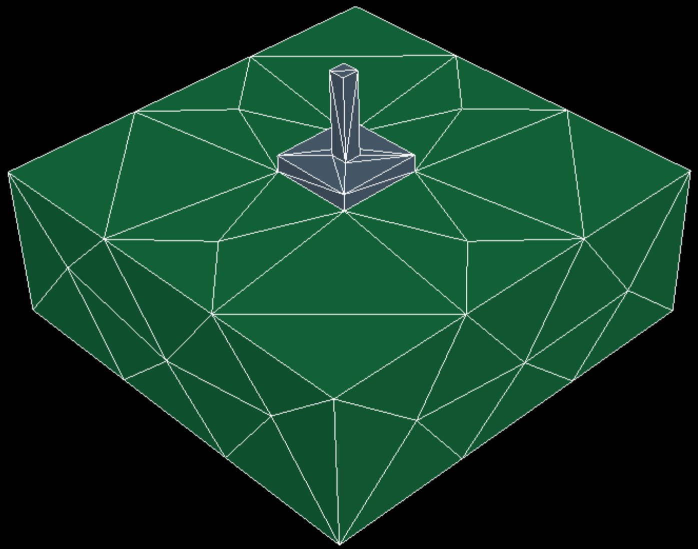

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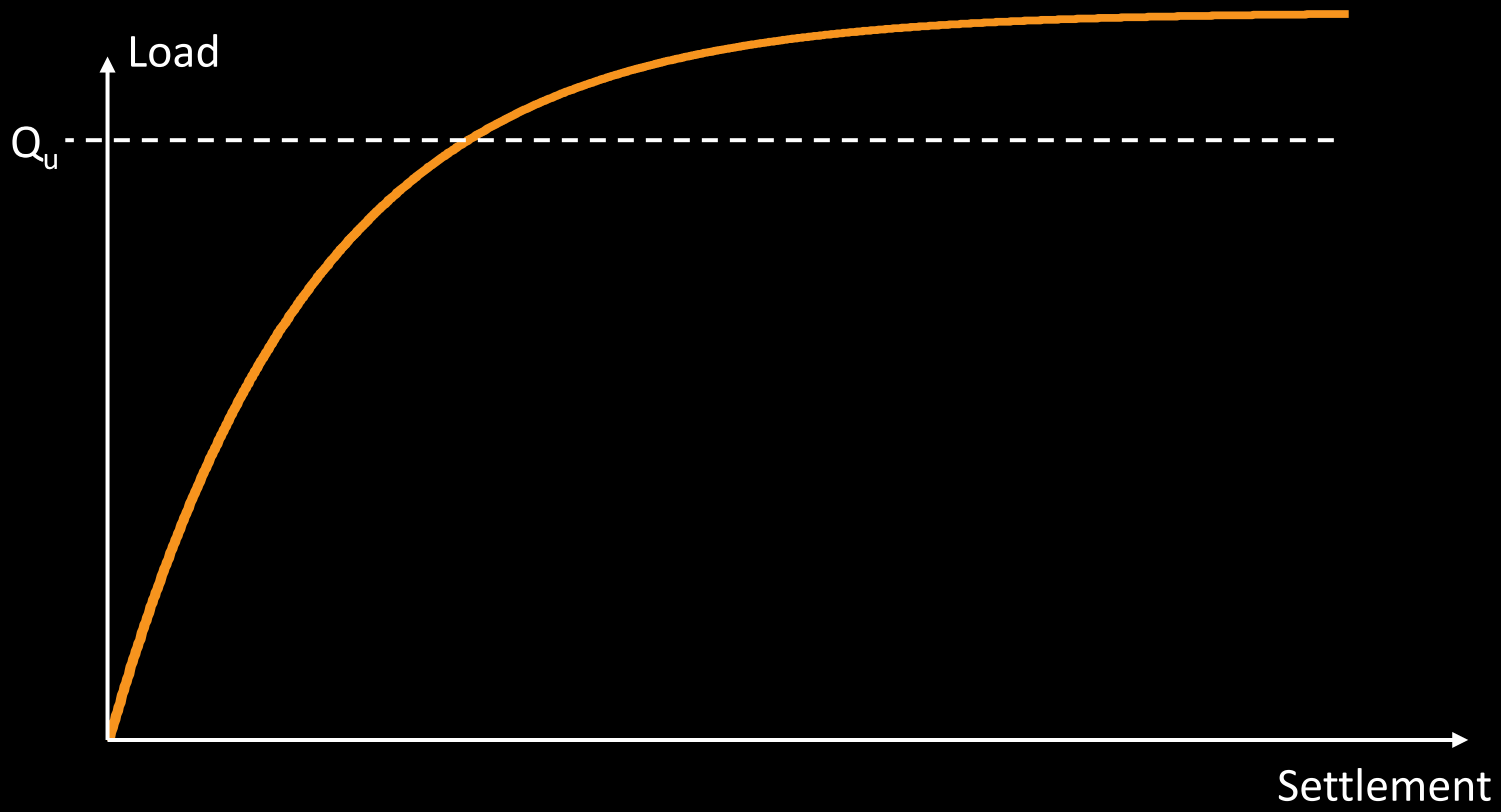

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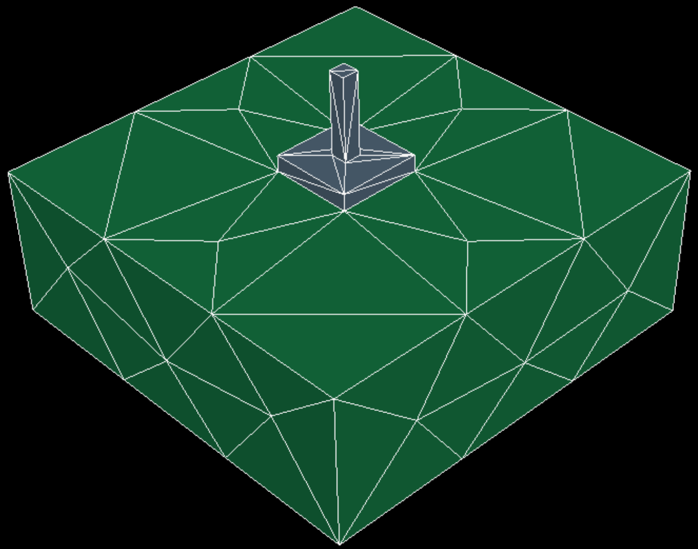

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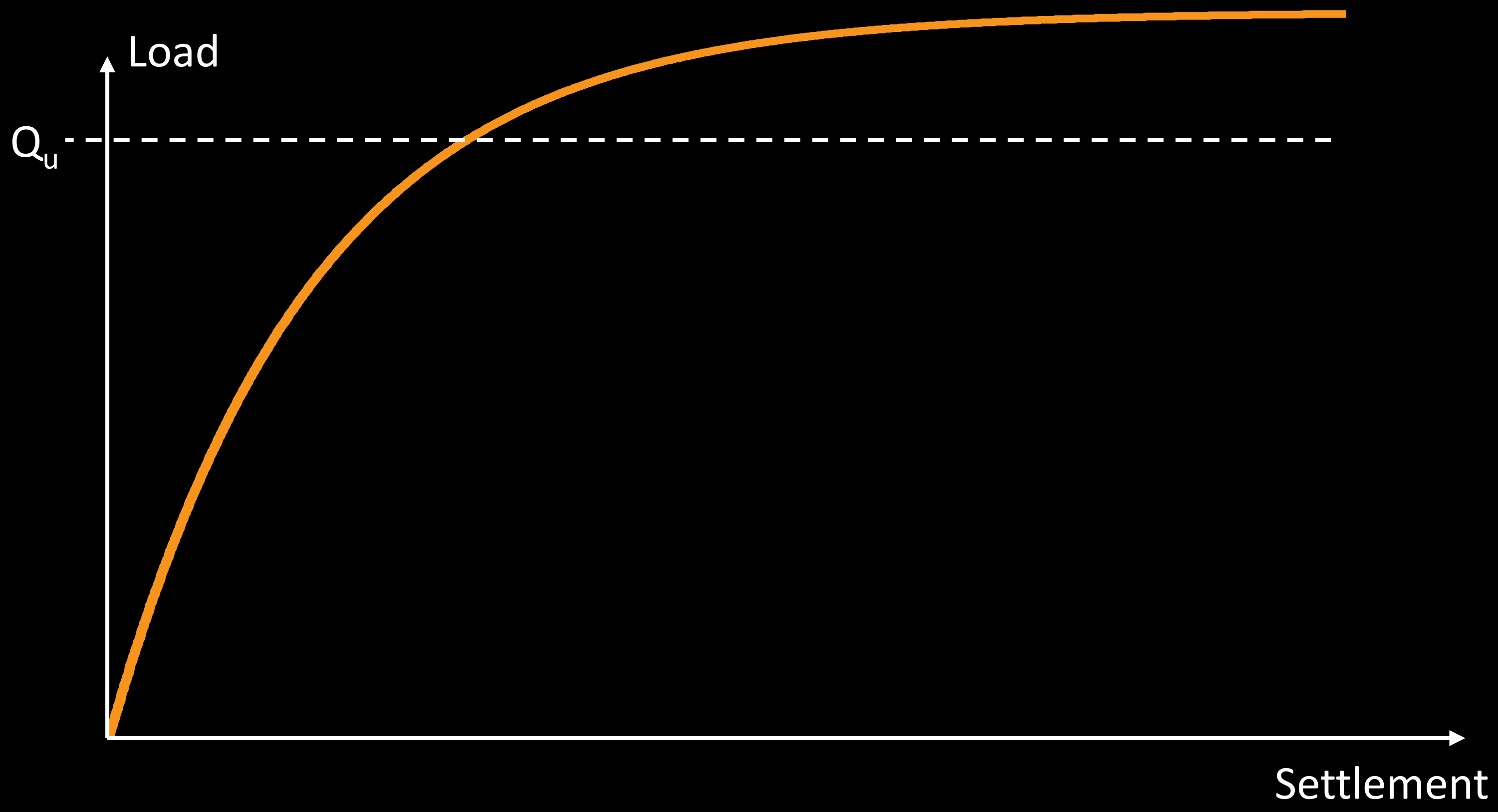


Is there a better way?

The Optum approach

E S S A I

Sur une application des règles de Maximis & Minimis *à quelques Problèmes de Statique, relatifs à l'Architecture.*

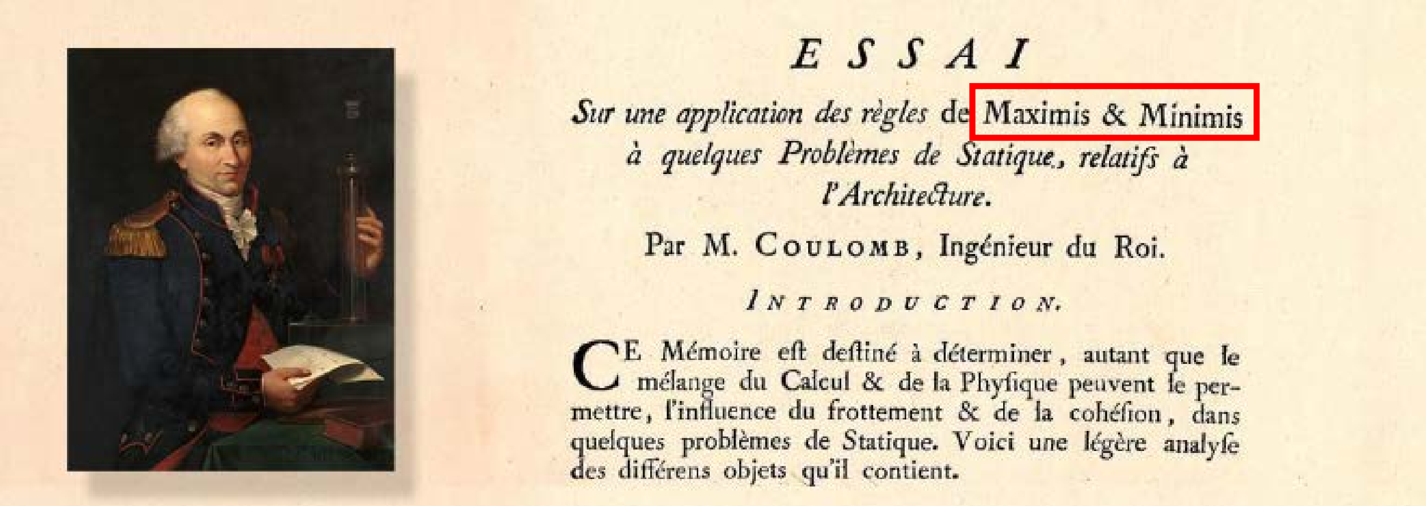
Par M. COULOMB, Ingénieur du Roi.

INTRODUCTION.

CE Mémoire eſt deſtiné à déterminer, autant que le mélange du Calcul & de la Phyſique peuvent le permettre, l'influence du frottement & de la cohéſion, dans quelques problèmes de Statique. Voici une légère analyſe des différens objets qu'il contient.

Coulomb (1736-1806)

The Optum approach



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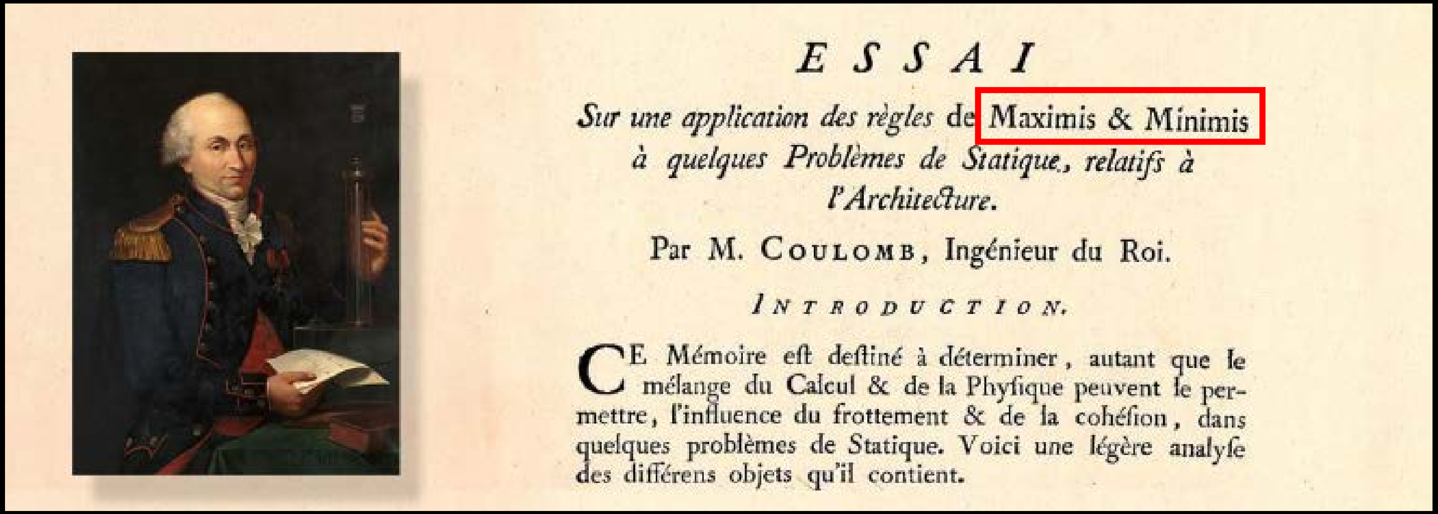

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Euler (1707-1783)

Since the fabric of the universe is most perfect and the work of a most wise Creator, nothing whatsoever takes place in the universe in which some relation of maximum or minimum does not appear.

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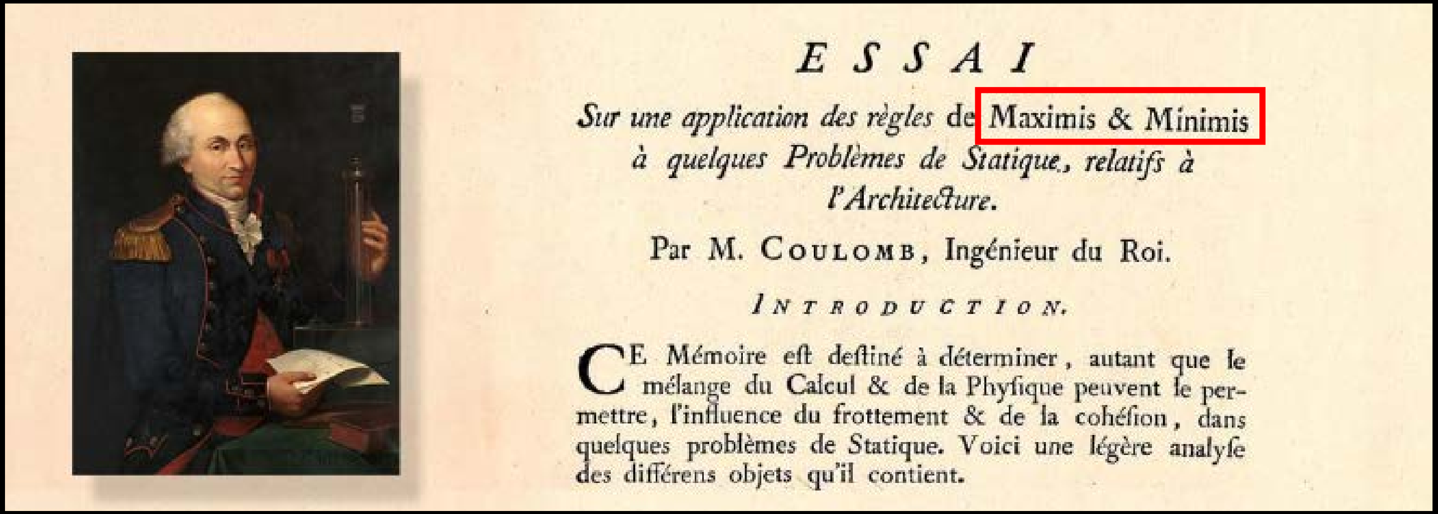
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Variational principles – examples:

- Fermat's principle (implying Snell's law of refraction)
- Classical mechanics (Maupertuis, Lagrange, Hamiltion)
- Principle of least action (classical and quantum mechanics)
- Hilbert's action principle (yielding Einstein's field equations)
- Mechanics of solids – elastostatics (e.g. Hellinger-Reissner), dynamics, plasticity and much more

Since the fabric of the universe is most perfect and the work of a most wise Creator, nothing whatsoever takes place in the universe in which some relation of maximum or minimum does not appear.

The Optum approach

Rigid plasticity (limit analysis)

Equilibrium: $\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0}$

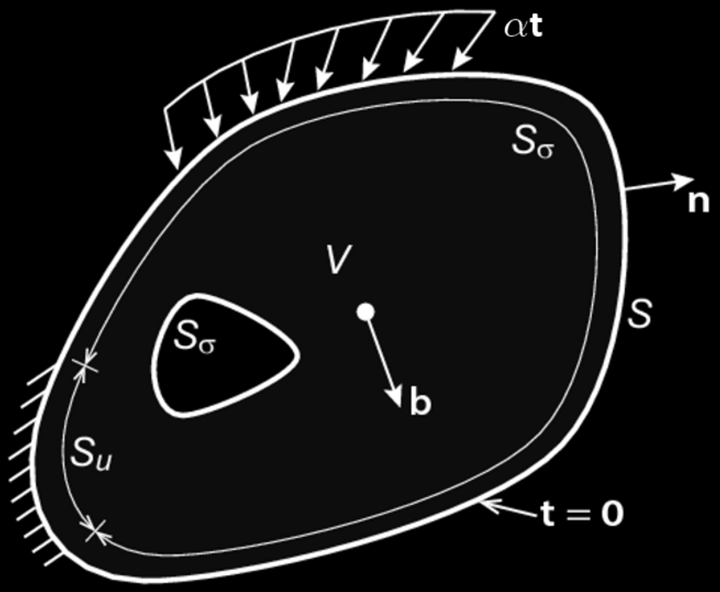
Static BC: $\boldsymbol{n} \cdot \boldsymbol{\sigma} = \alpha \boldsymbol{t}$ on S_σ

Strain-disp: $\dot{\boldsymbol{\varepsilon}}^p = \nabla \dot{\boldsymbol{u}}$

Yield condition: $F(\boldsymbol{\sigma}) \leq 0$

Flow rule: $\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \dfrac{\partial F}{\partial \boldsymbol{\sigma}}$

Complementarity: $\dot{\lambda} \geq 0, \quad \dot{\lambda} F(\boldsymbol{\sigma}) = 0$



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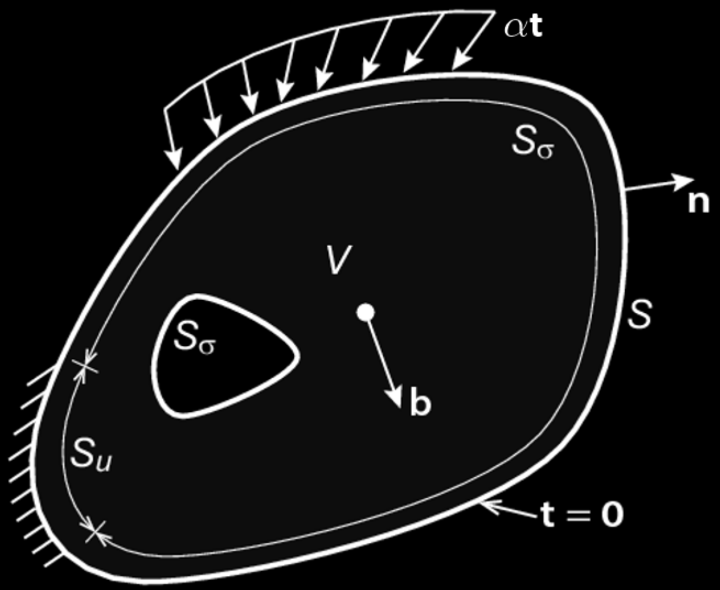

Variational principle (lower bound):

maximize α

subject to $\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0}$

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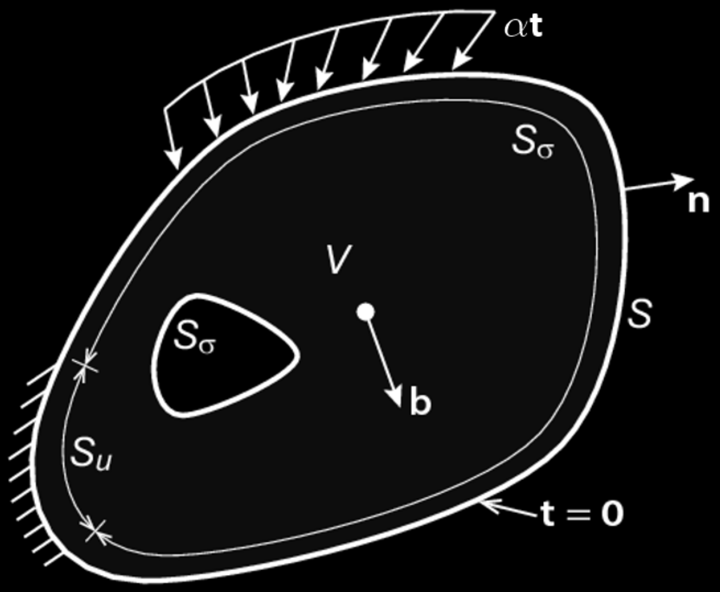

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The variational principle (optimization problem) is in every way equivalent to the governing equations

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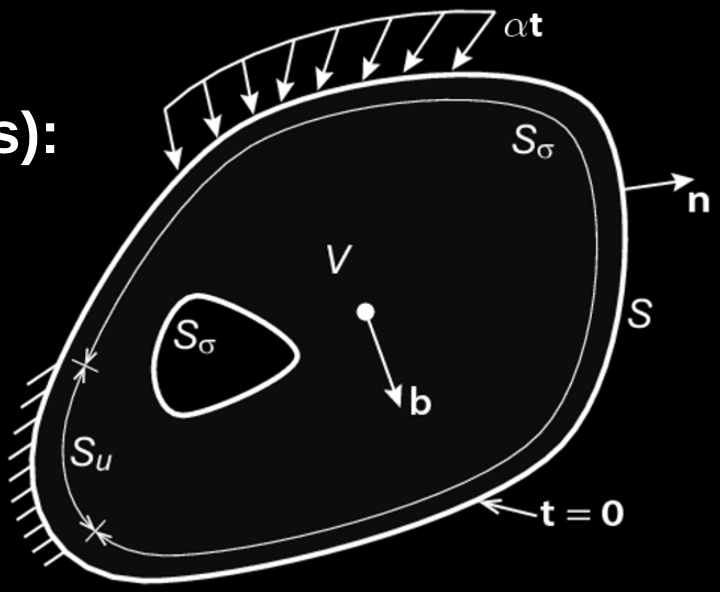

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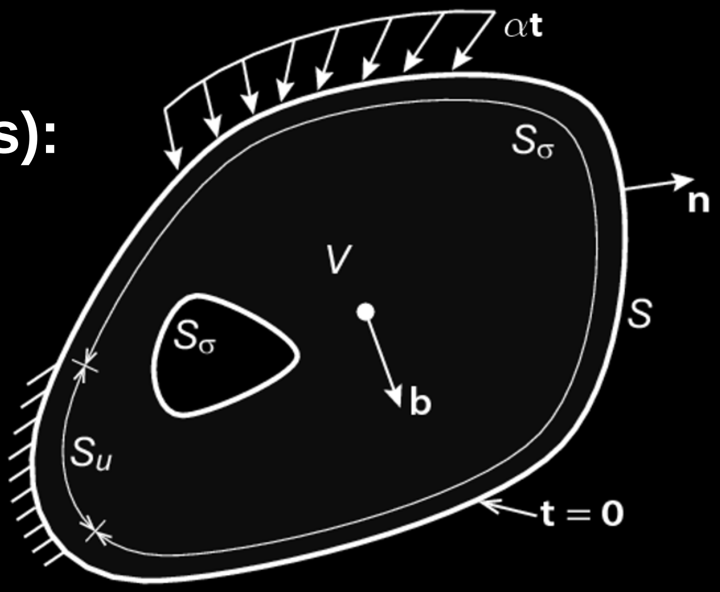

Simple analogy: Hooke's law

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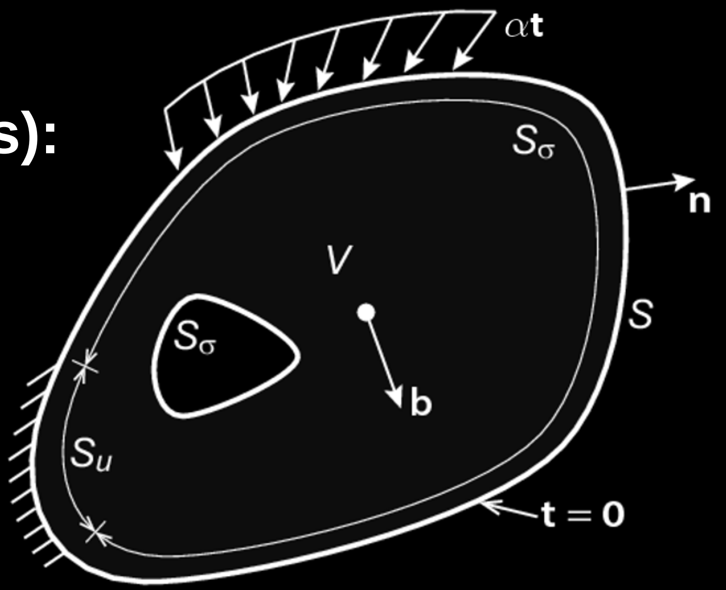

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Either:

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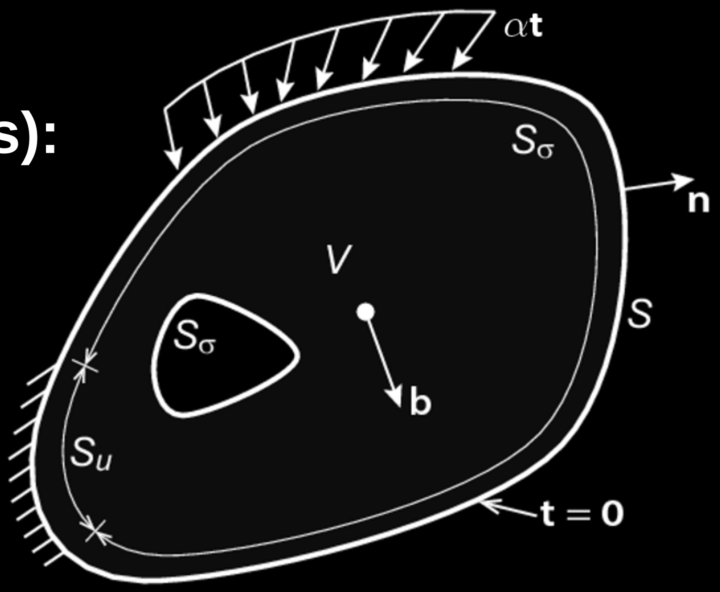

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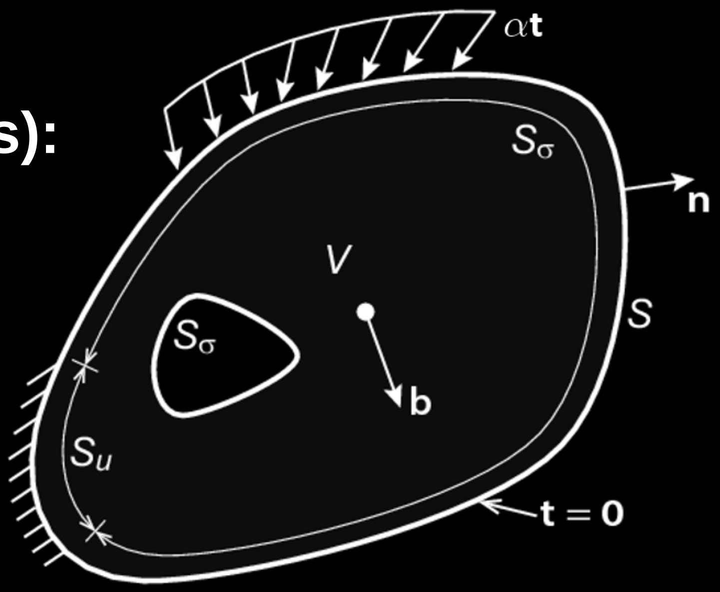

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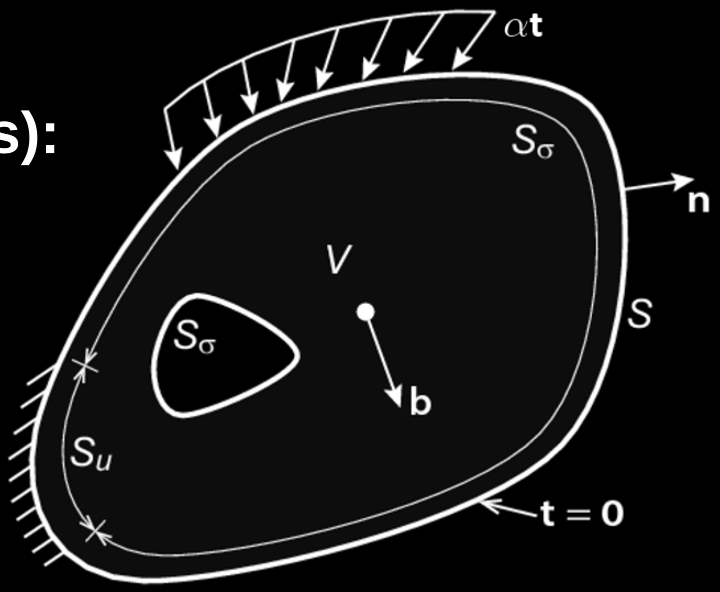

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Rigid plasticity (limit analysis)

Equilibrium: $\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0}$

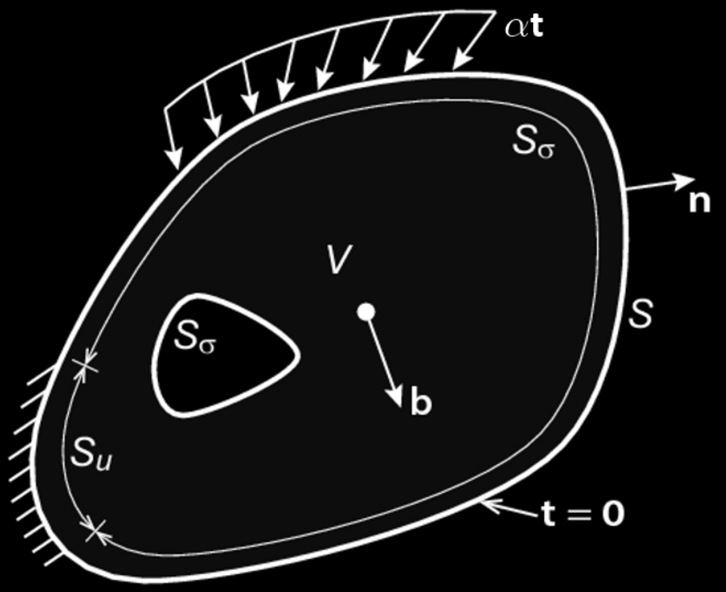
Static BC: $\boldsymbol{n} \cdot \boldsymbol{\sigma} = \alpha \boldsymbol{t}$ on S_σ

Strain-disp: $\dot{\boldsymbol{\varepsilon}}^p = \nabla \dot{\boldsymbol{u}}$

Yield condition: $F(\boldsymbol{\sigma}) \leq 0$

Flow rule: $\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \dfrac{\partial F}{\partial \boldsymbol{\sigma}}$

Complementarity: $\dot{\lambda} \geq 0, \quad \dot{\lambda} F(\boldsymbol{\sigma}) = 0$



Variational principle (lower bound):

maximize α

subject to $\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0}$

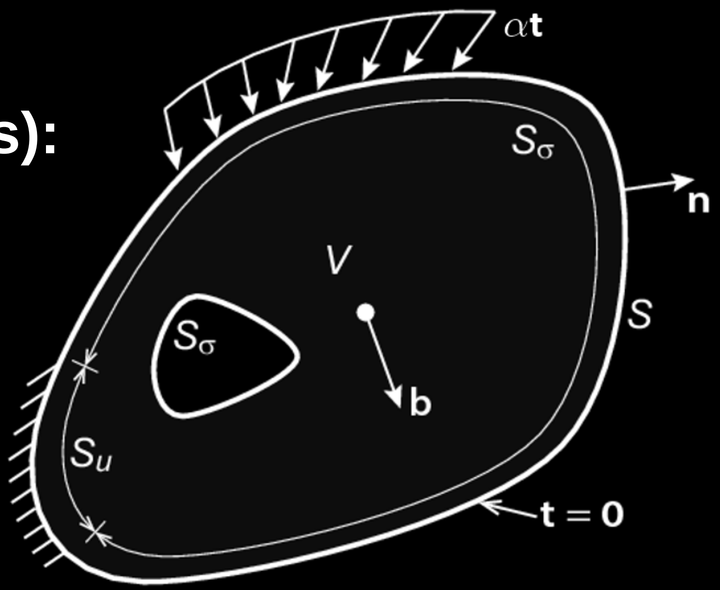
$\boldsymbol{n} \cdot \boldsymbol{\sigma} = \alpha \boldsymbol{t}$ on S_σ

$F(\boldsymbol{\sigma}) \leq 0$

The variational principle (optimization problem) is in every way equivalent to the governing equations

The Optum approach

Variational principle (equivalent to governing equations):

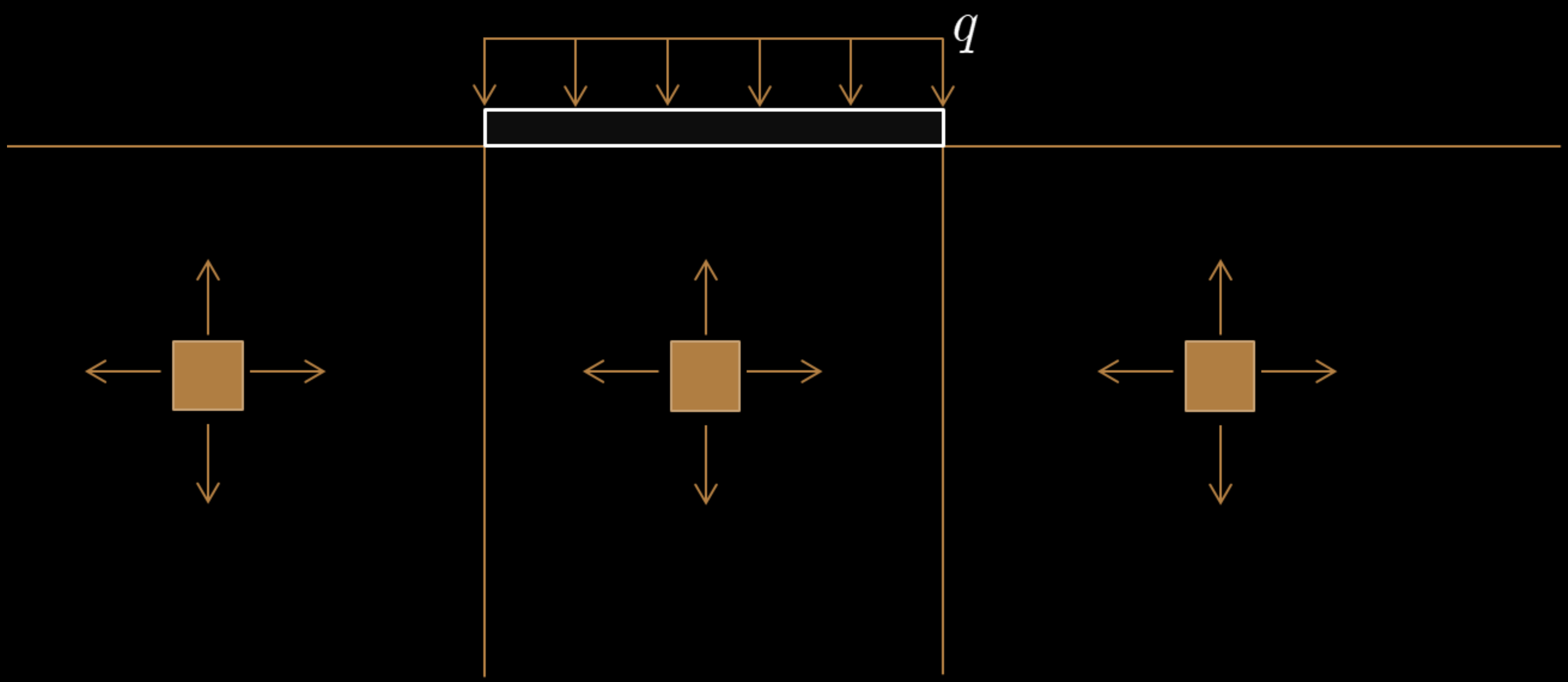


$$\begin{aligned} \text{maximize} \quad & \alpha \\ \text{subject to} \quad & \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0} \\ & \boldsymbol{n} \cdot \boldsymbol{\sigma} = \alpha \boldsymbol{t} \quad \text{on } S_\sigma \\ & F(\boldsymbol{\sigma}) \leq 0 \end{aligned}$$

However: partial solutions (solutions that satisfy the constraints but are not optimal) will be lower bounds (less than the optimal α)

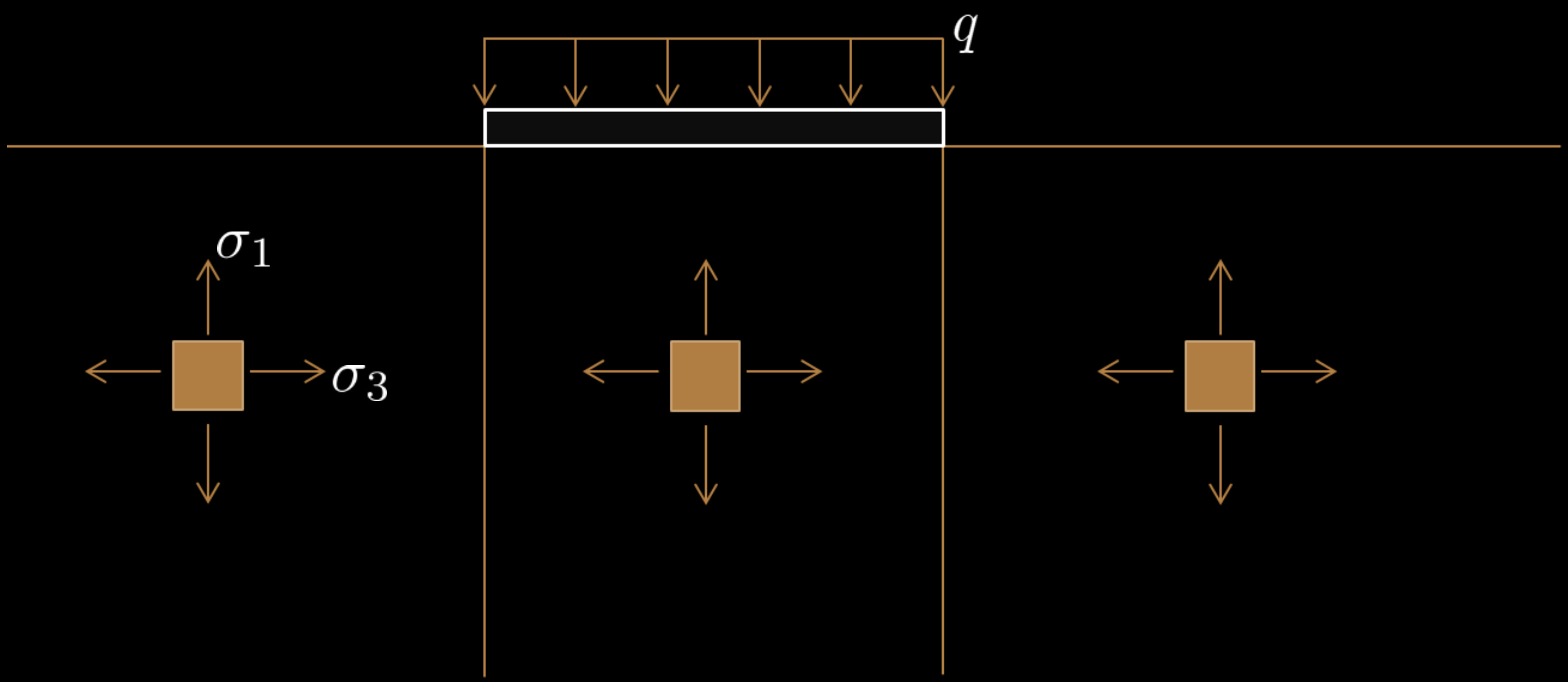
The Optum approach

Example: footing on weightless Tresca soil



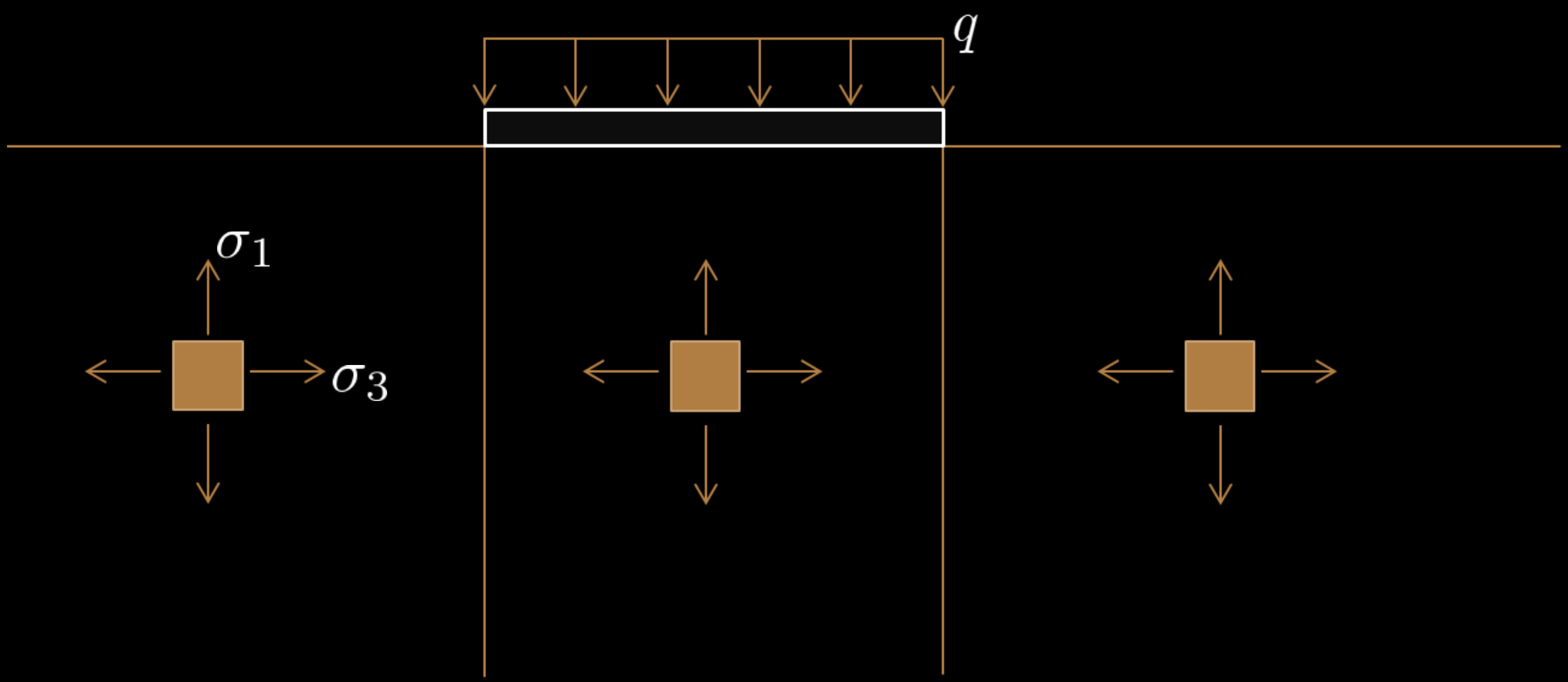
The Optum approach

Example: footing on weightless Tresca soil



The Optum approach

Example: footing on weightless Tresca soil

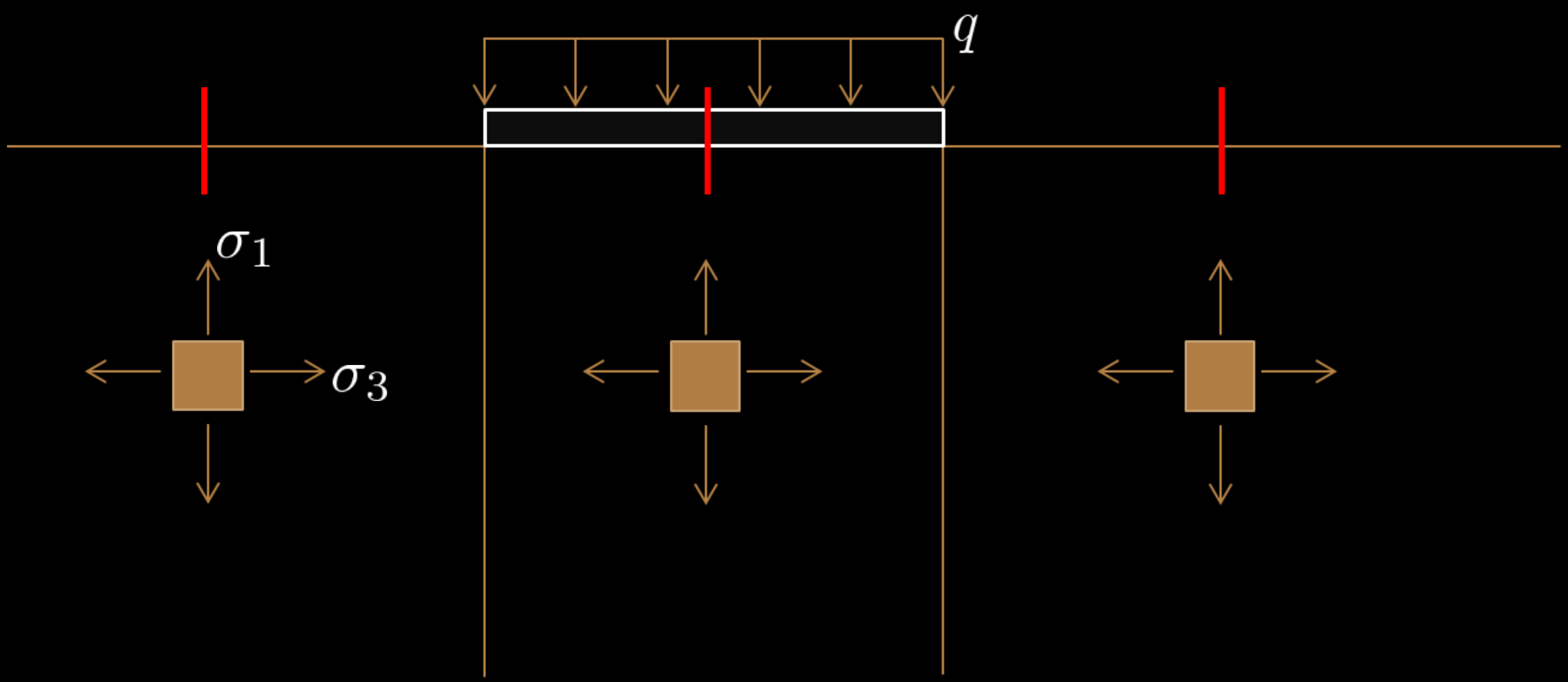


maximize q

subject to
- Global equilibrium (satisfied)
- Static BCs
- Interface equilibrium
- $|\sigma_1 - \sigma_3| \leq 2s_u$

The Optum approach

Example: footing on weightless Tresca soil



maximize q

subject to Global equilibrium (satisfied)

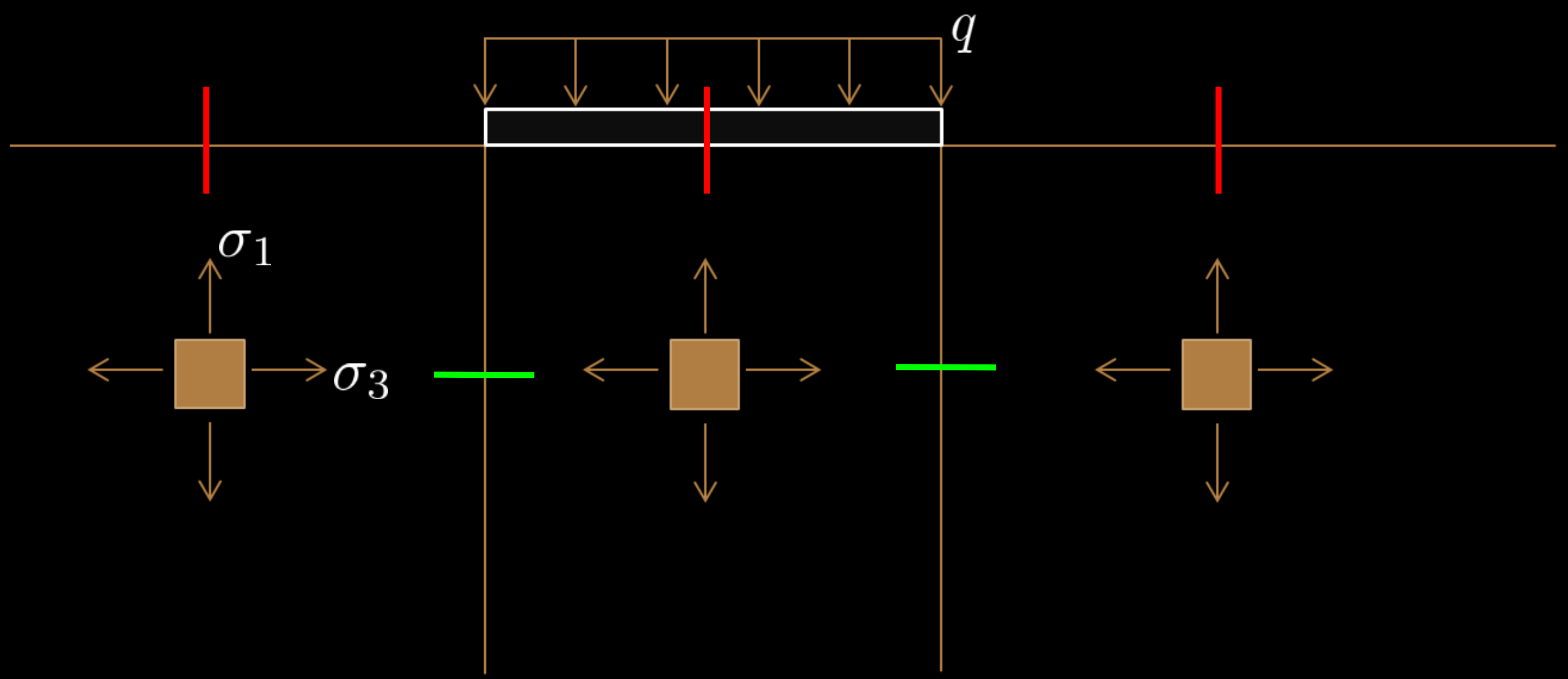
Static BCs

Interface equilibrium

$|\sigma_1 - \sigma_3| \leq 2s_u$

The Optum approach

Example: footing on weightless Tresca soil



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The Optum approach

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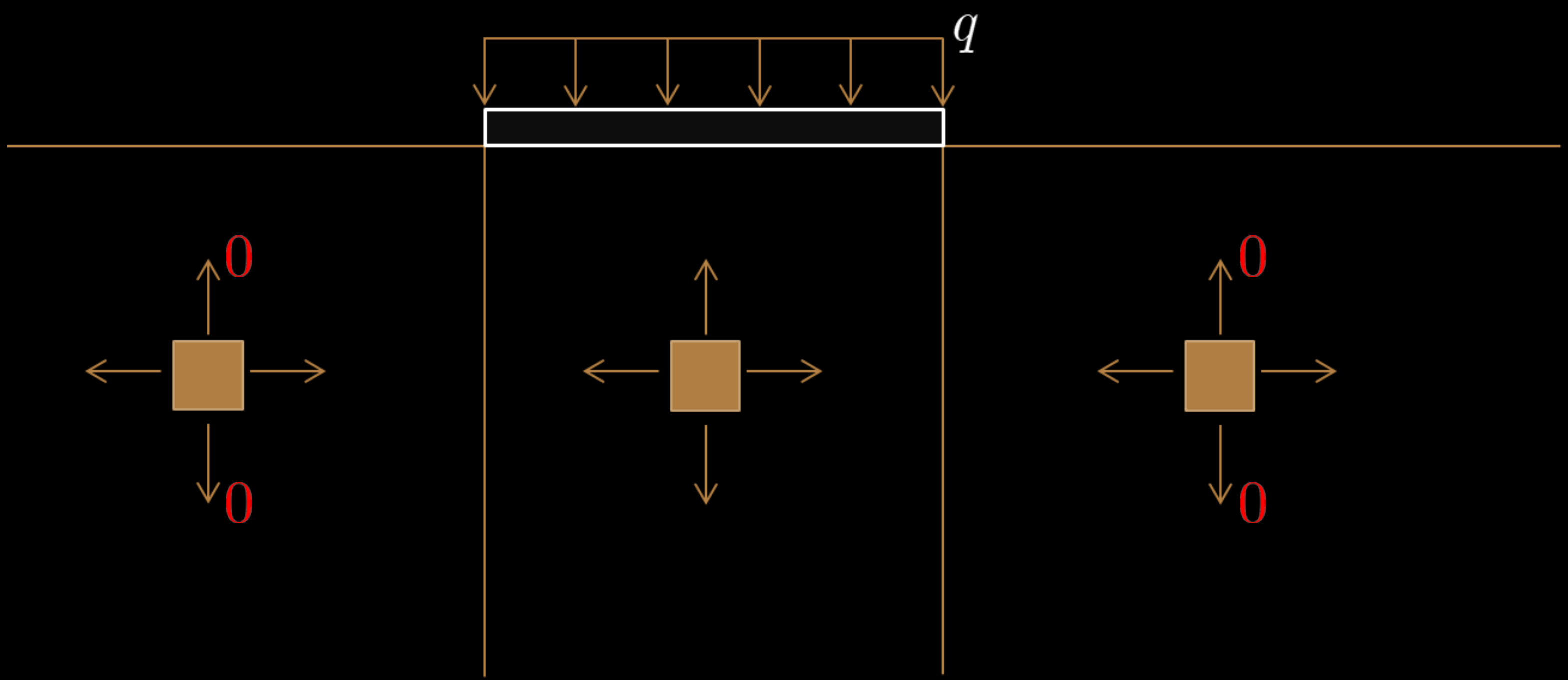

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The Optum approach

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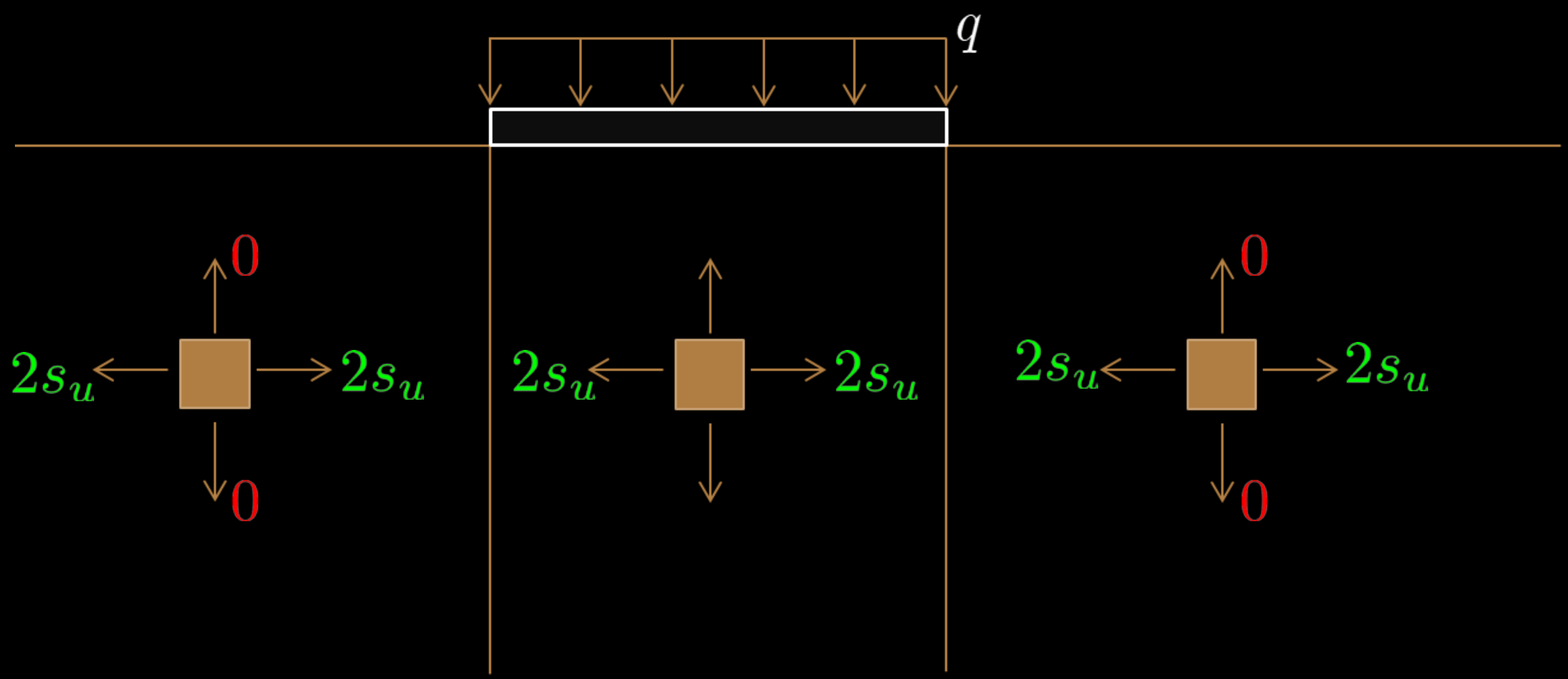

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Static BCs

Interface equilibrium

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The Optum approach

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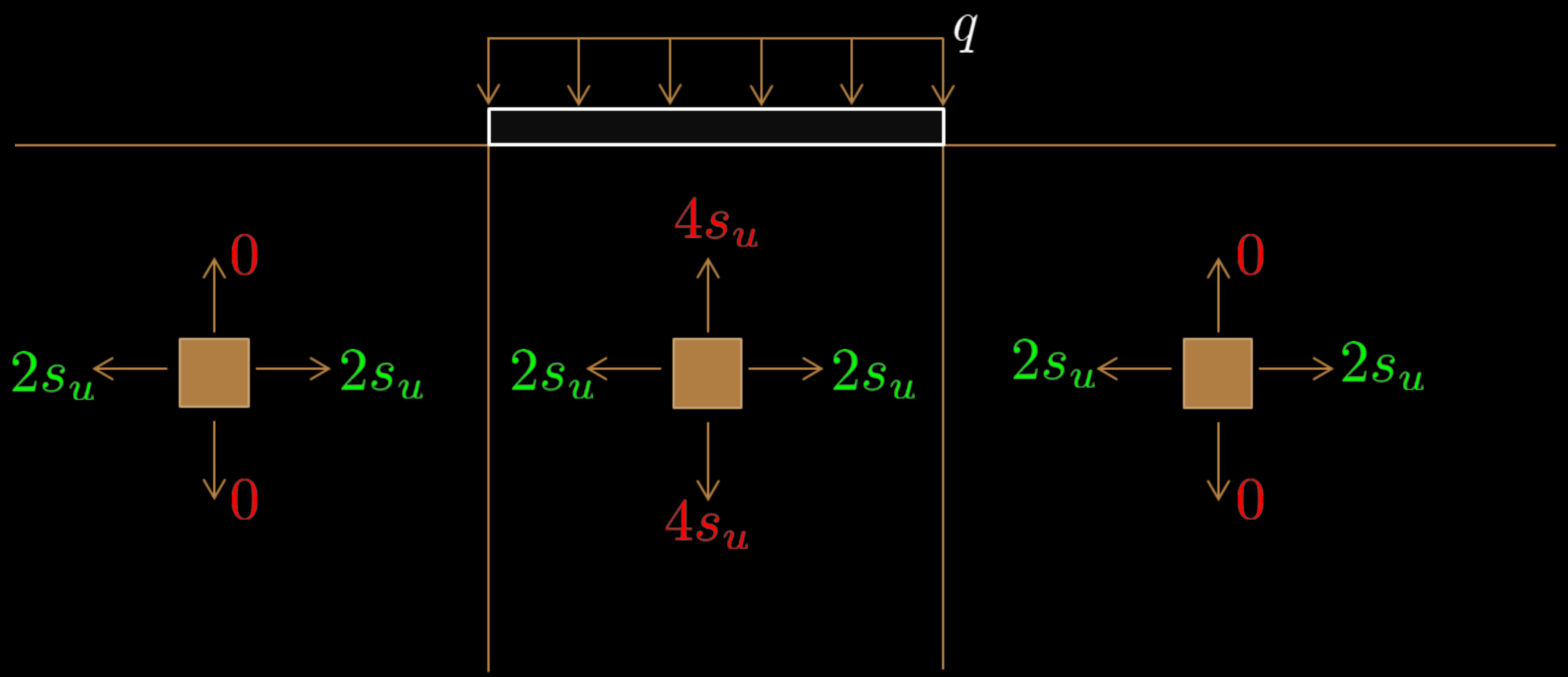

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Static BCs

Interface equilibrium

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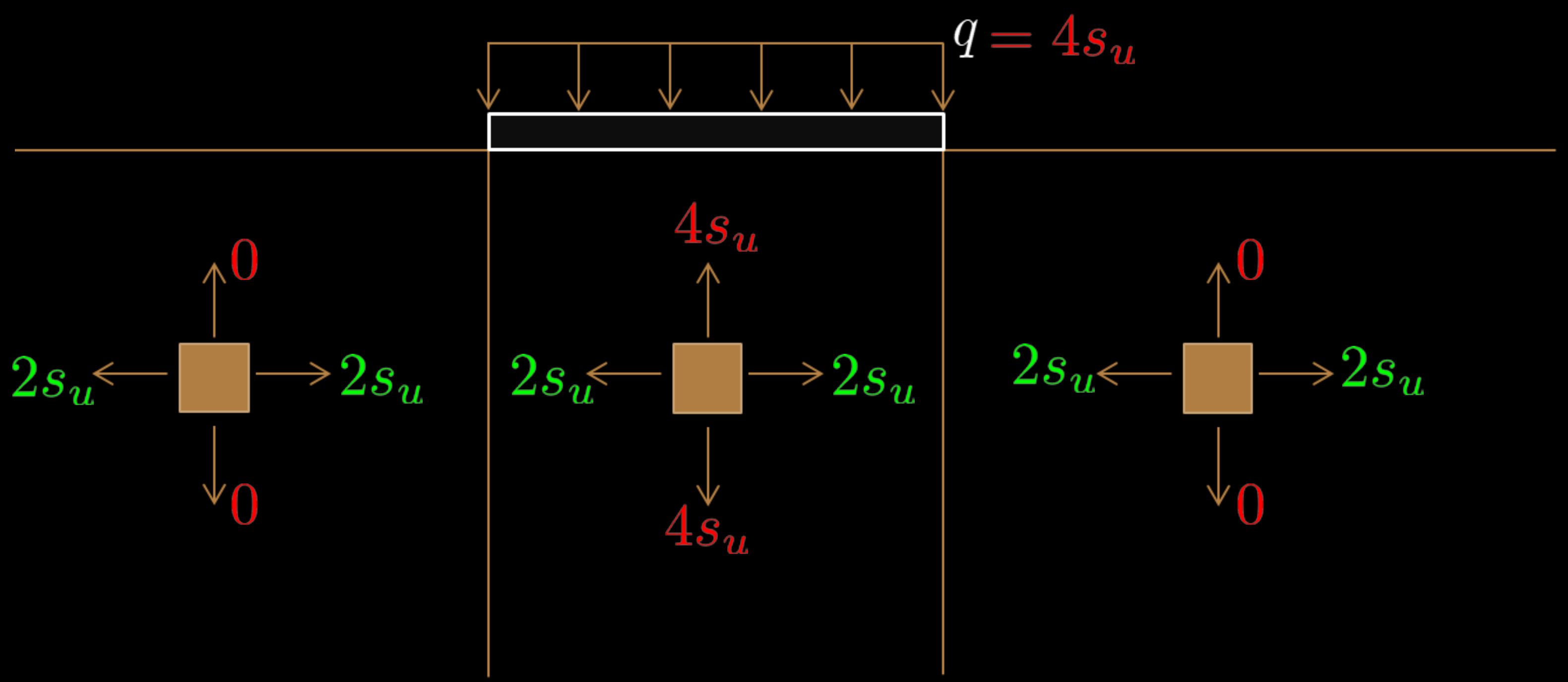
The Optum approach

Example: footing on weightless Tresca soil



maximize q

subject to Global equilibrium (satisfied)

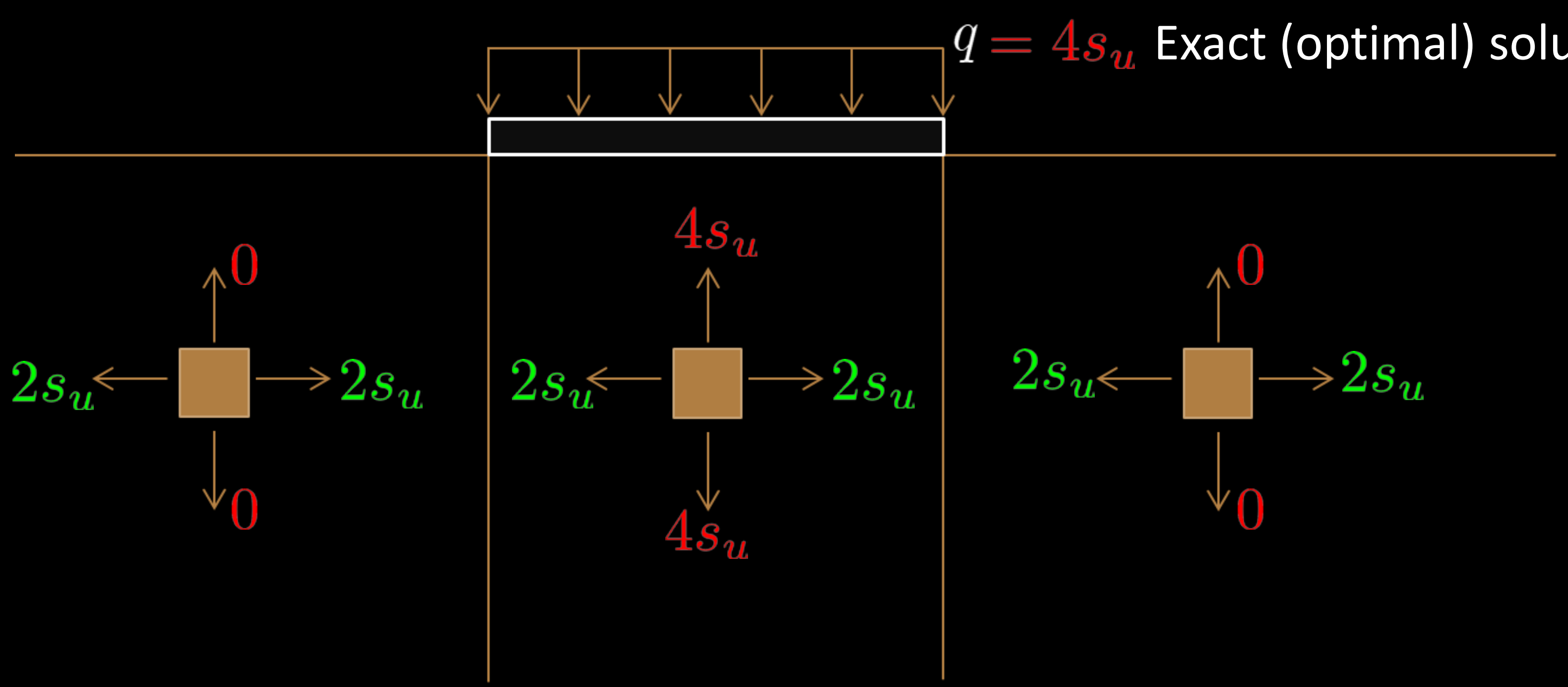
Static BCs

Interface equilibrium

$|\sigma_1 - \sigma_3| \leq 2s_u$

The Optum approach

Example: footing on weightless Tresca soil



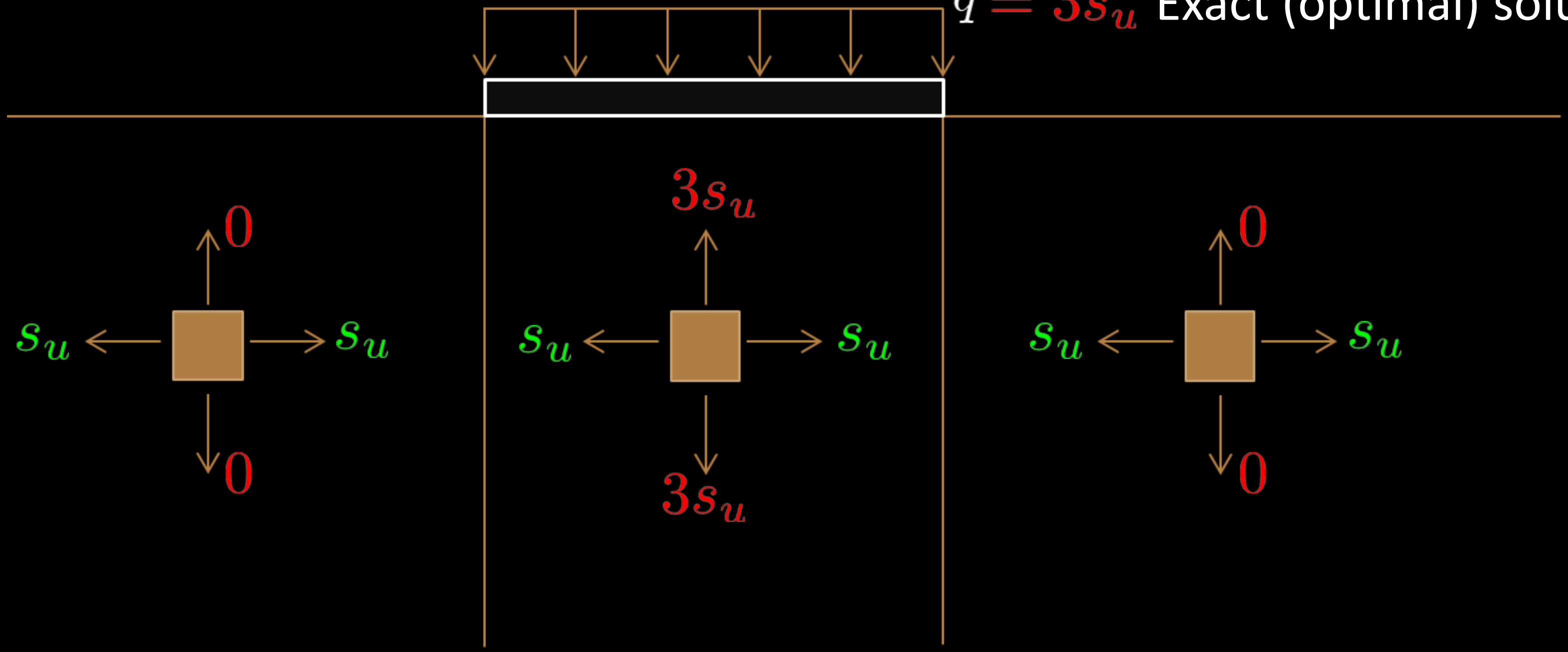
Exact (optimal) solution: $q = 5.14s_u$

maximize	q
subject to	Global equilibrium (satisfied)
	Static BCs
	Interface equilibrium
	$\lvert\sigma_1 - \sigma_3\rvert \leq 2s_u$

The Optum approach

Example: footing on weightless Tresca soil

$q = 3s_u$ Exact (optimal) solution: $q = 5.14s_u$



maximize q

subject to Global equilibrium (satisfied)

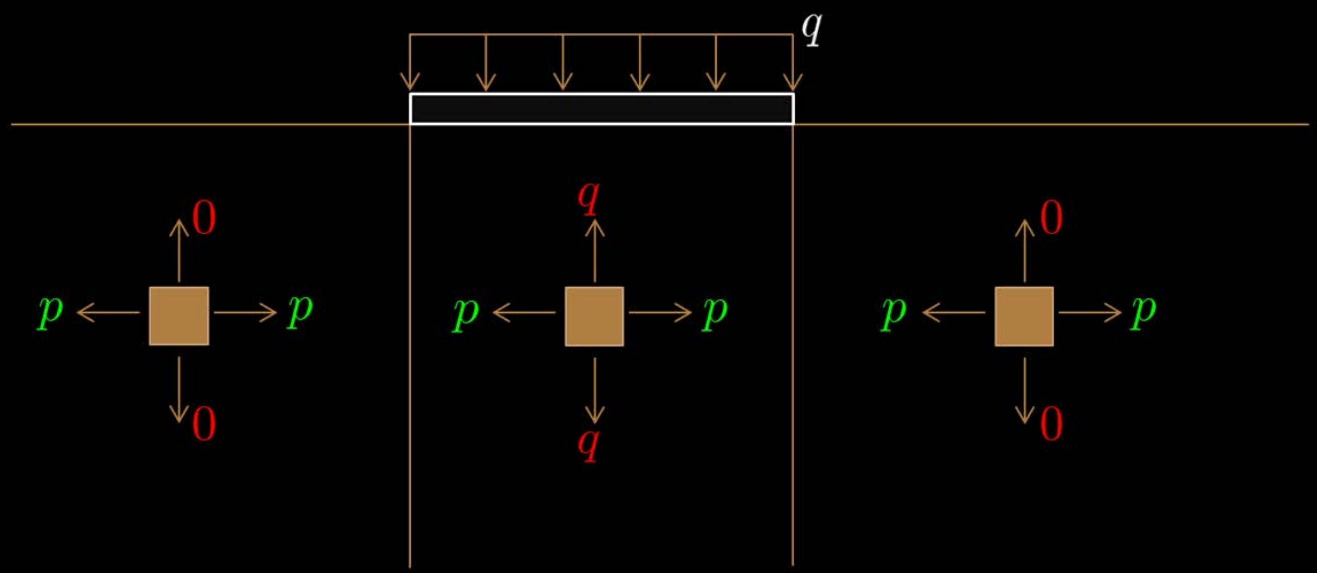
Static BCs

Interface equilibrium

$|\sigma_1 - \sigma_3| \leq 2s_u$

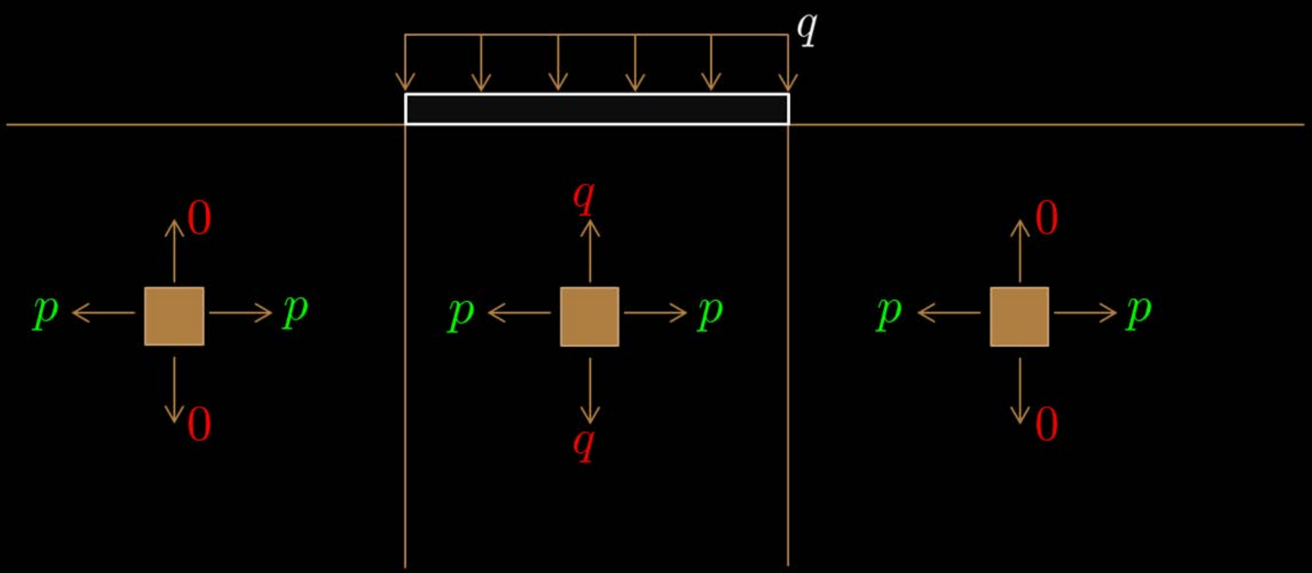
The Optum approach

Example: footing on weightless Tresca soil



The Optum approach

Example: footing on weightless Tresca soil



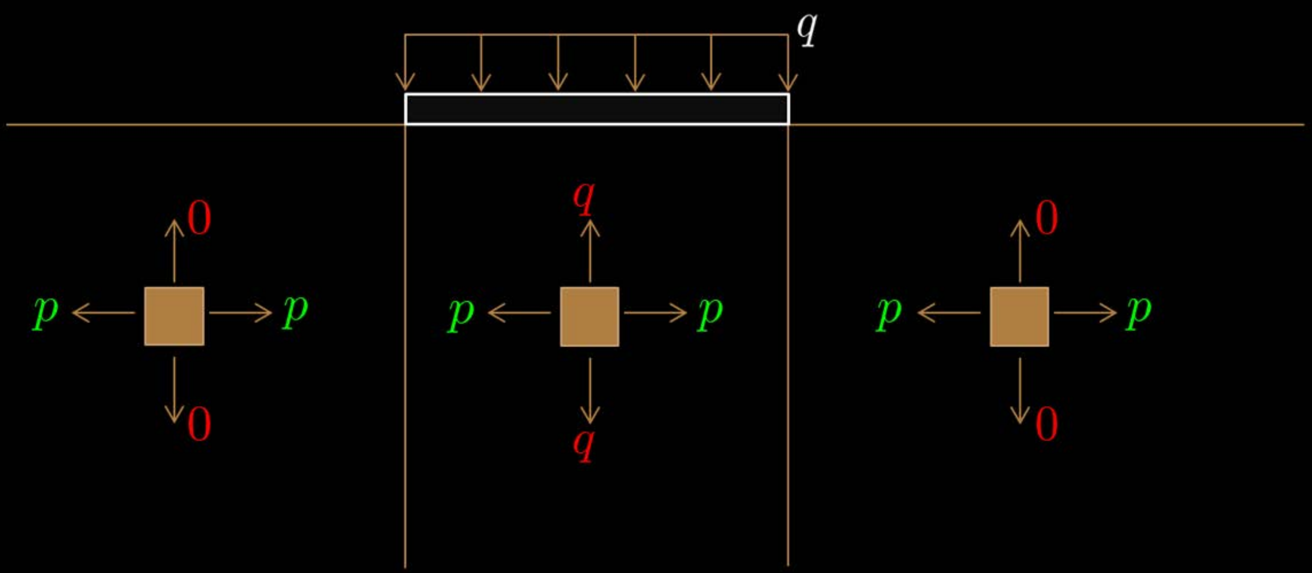
maximize q

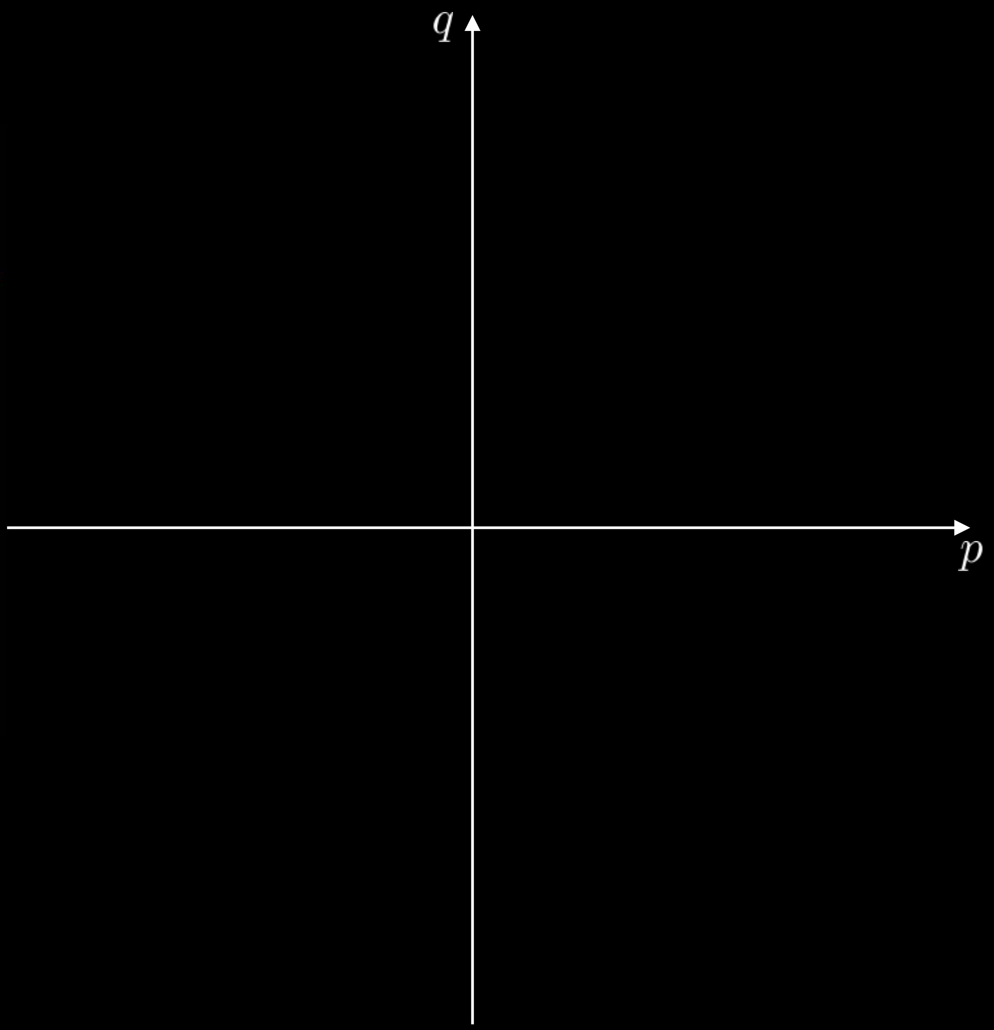
subject to $|p| \leq 2s_u$

$|q - p| \leq 2s_u$

The Optum approach

Example: footing on weightless Tresca soil





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The Optum approach

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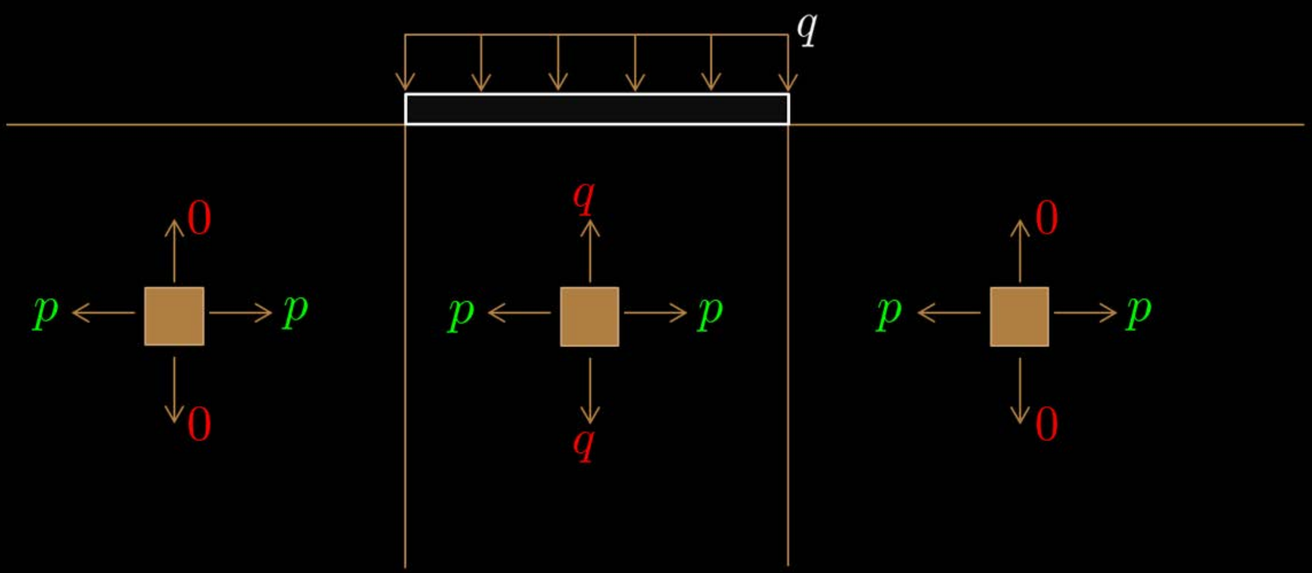

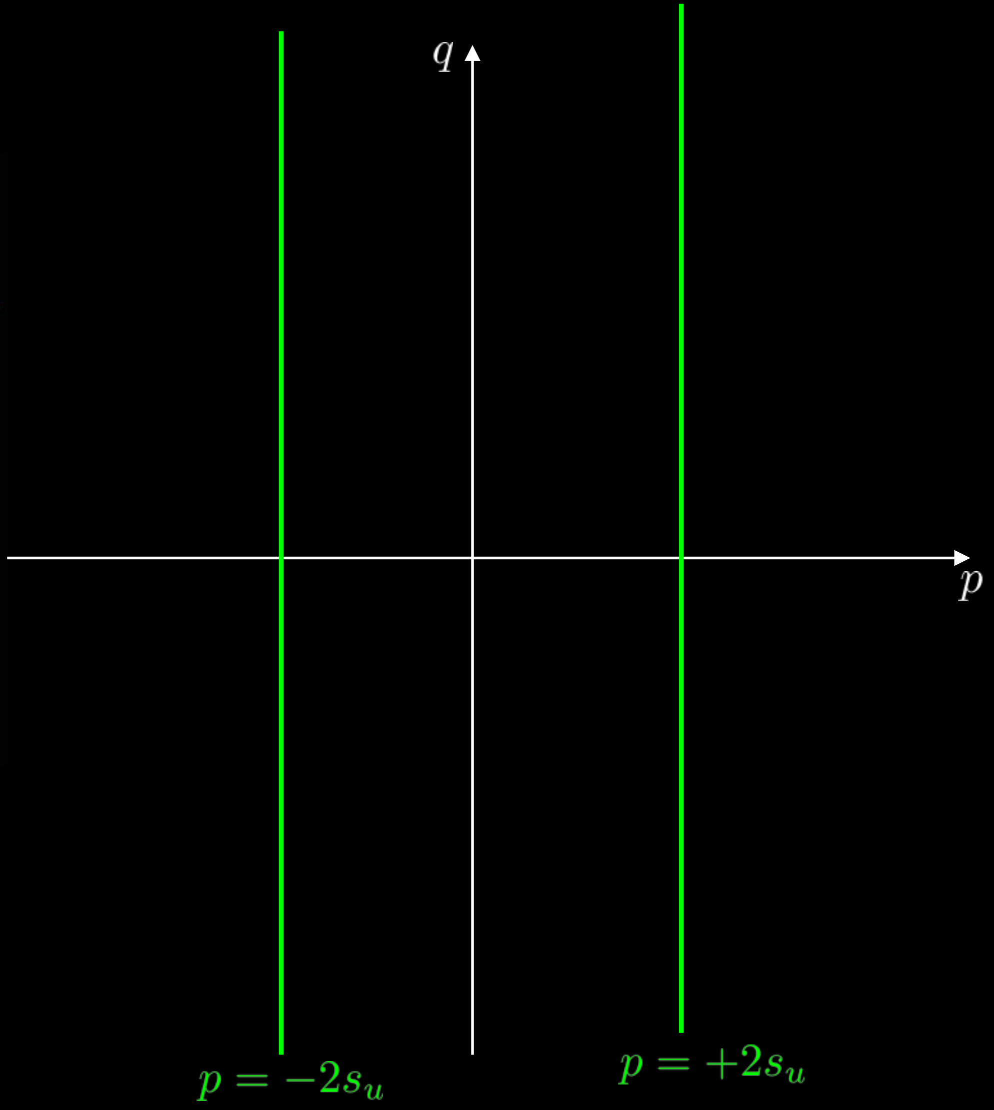

maximize q

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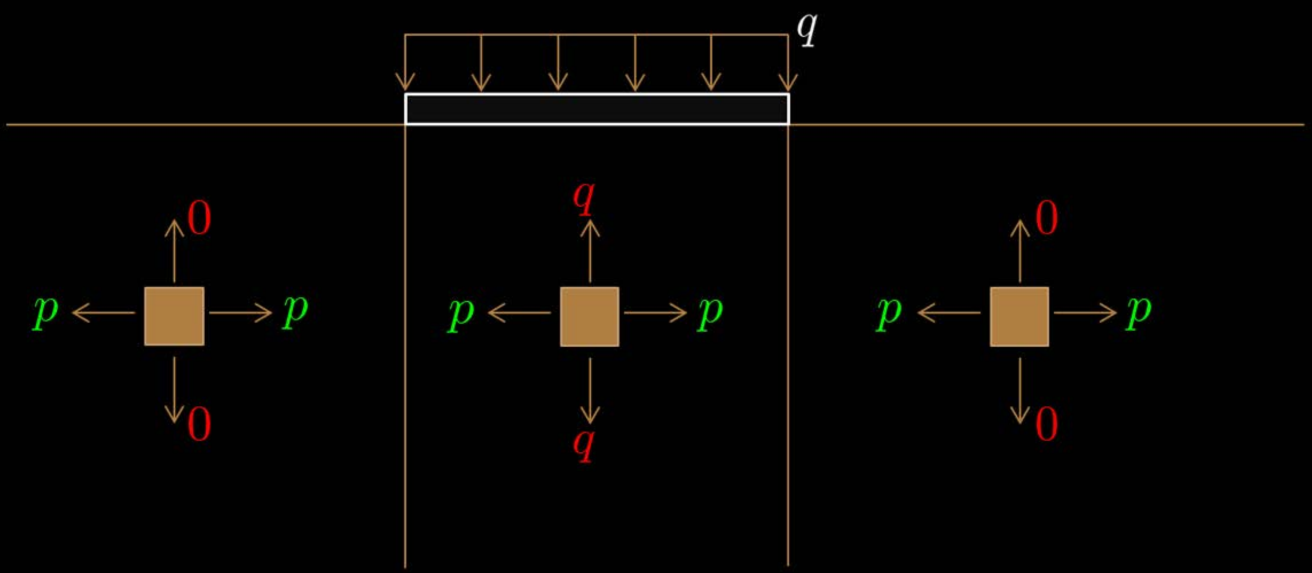
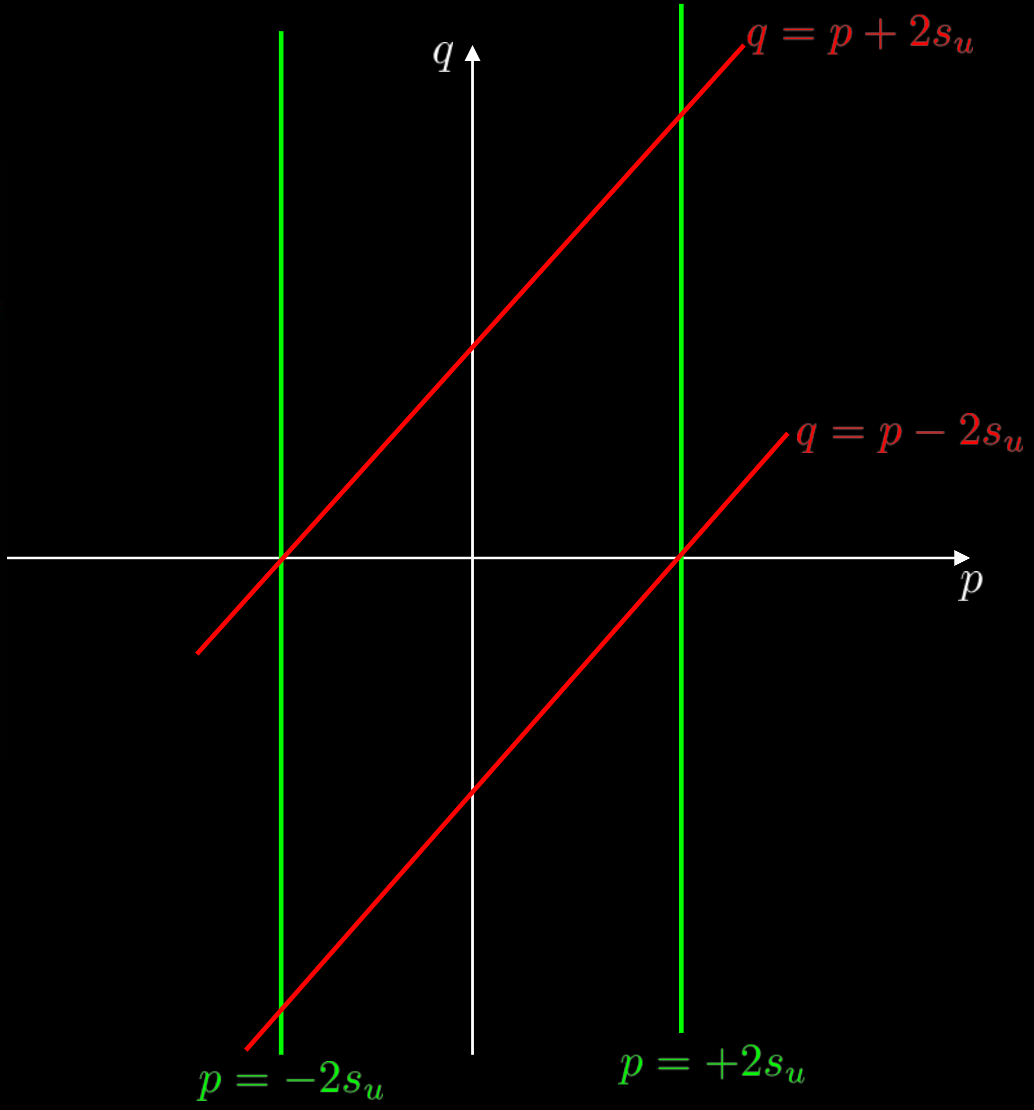
$|q - p| \leq 2s_u$

The Optum approach

Example: footing on weightless Tresca soil





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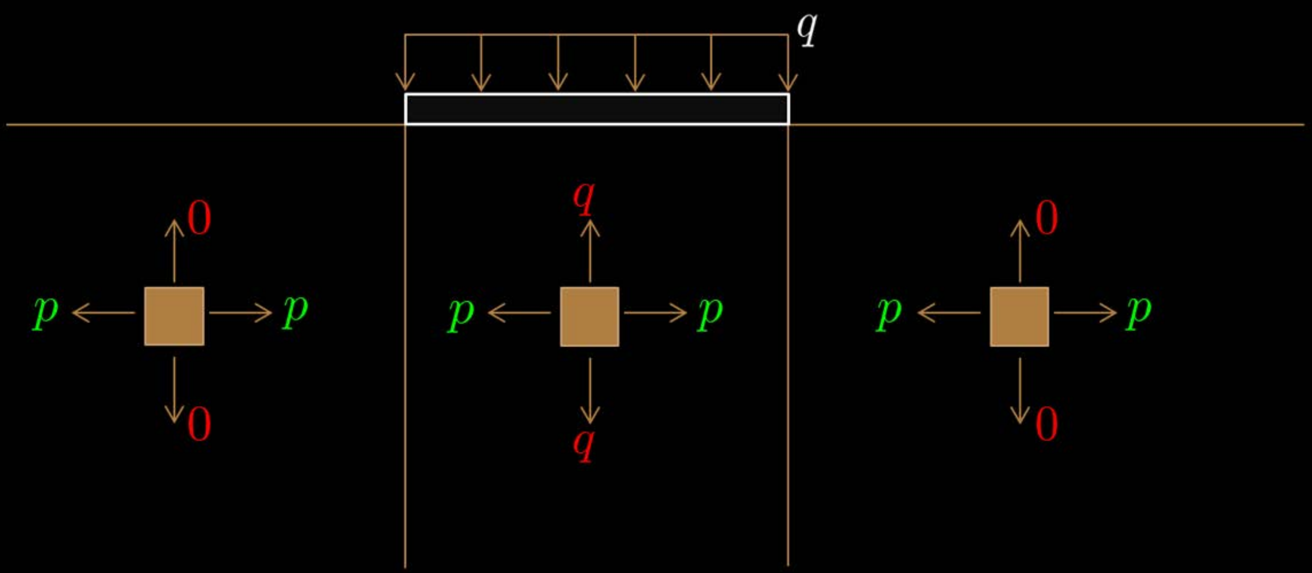
The Optum approach

Example: footing on weightless Tresca soil



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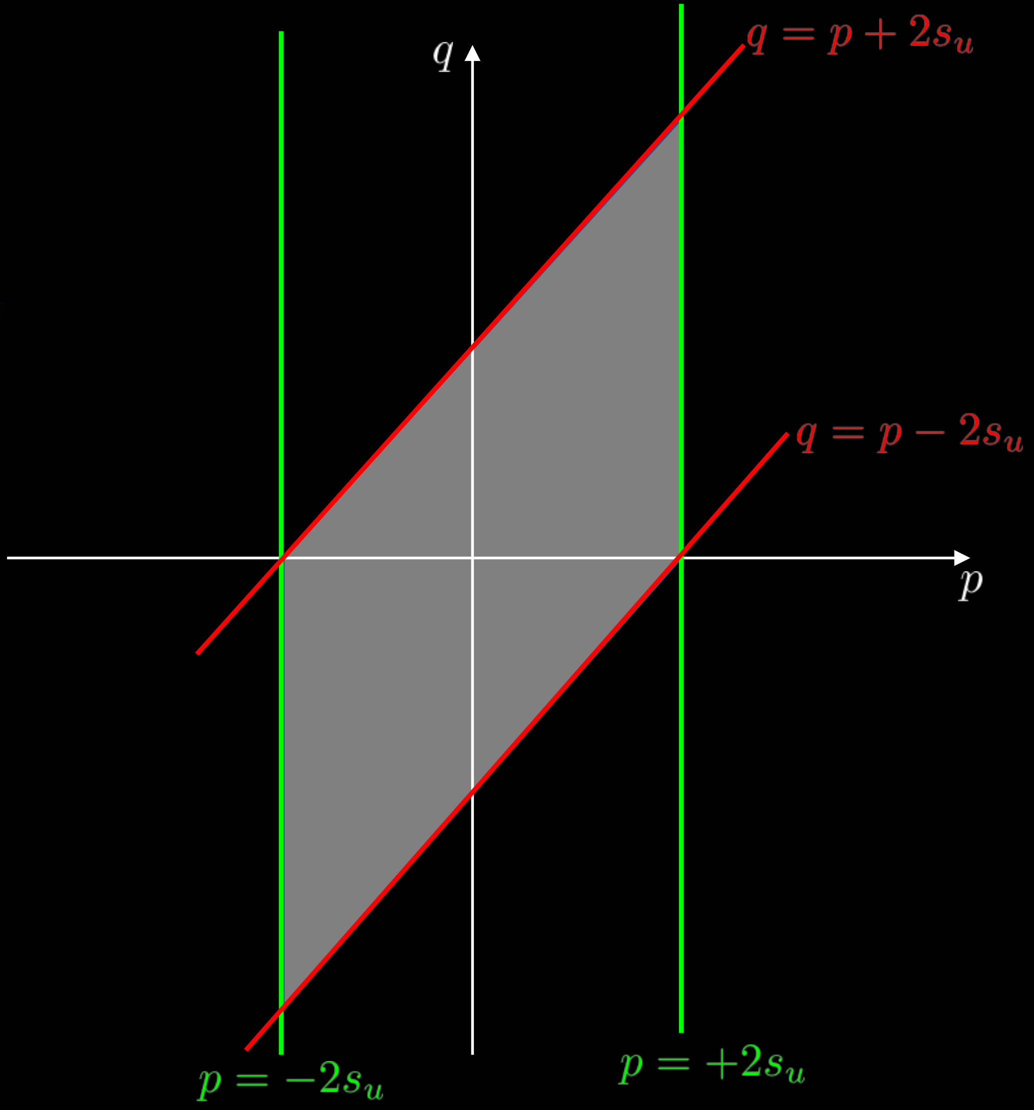
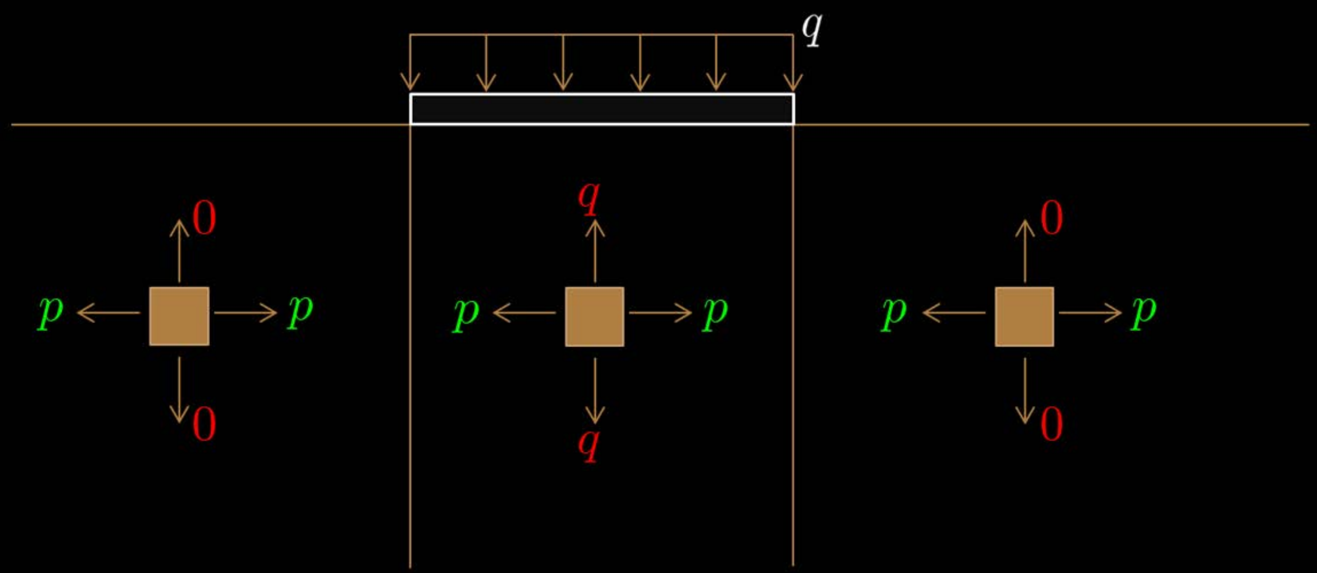


The Optum approach

Example: footing on weightless Tresca soil



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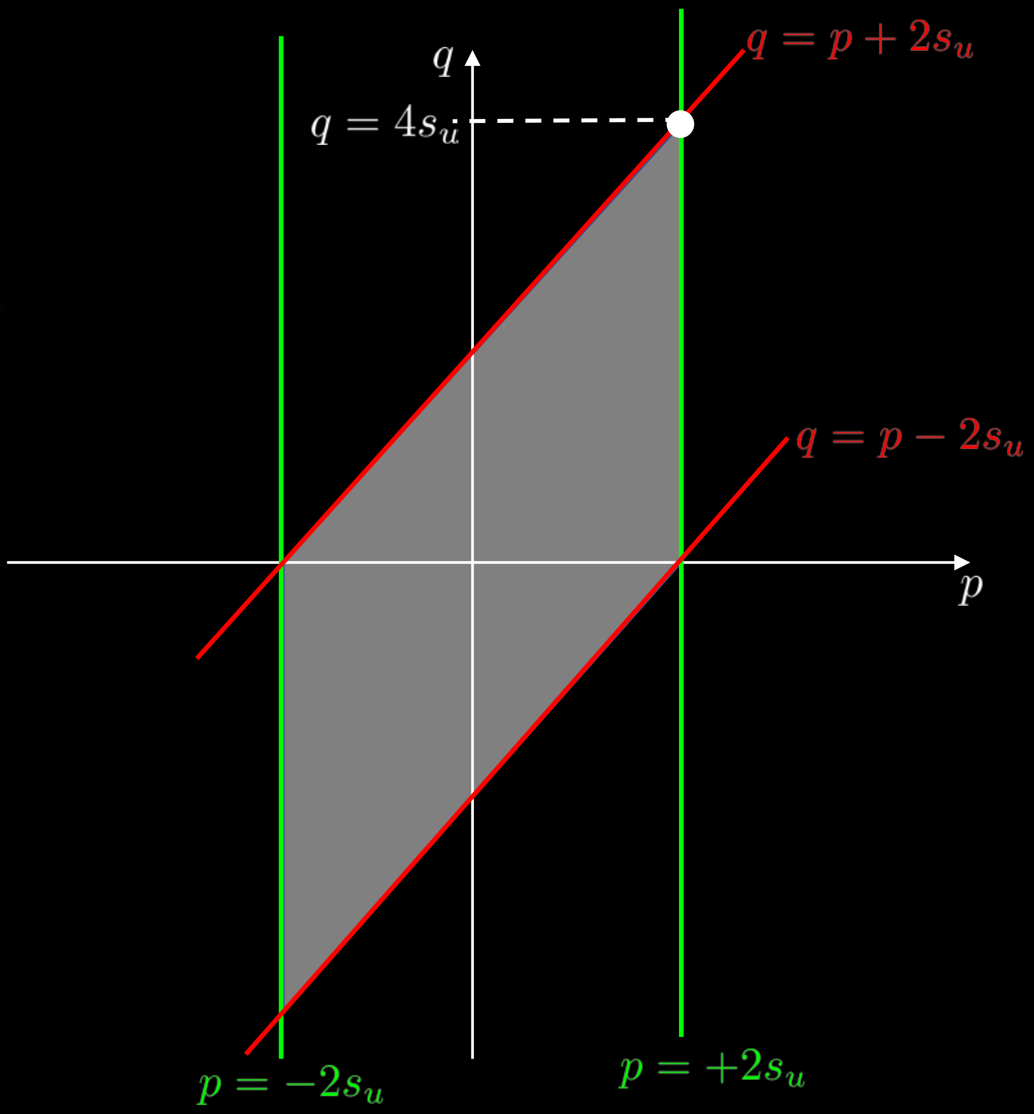
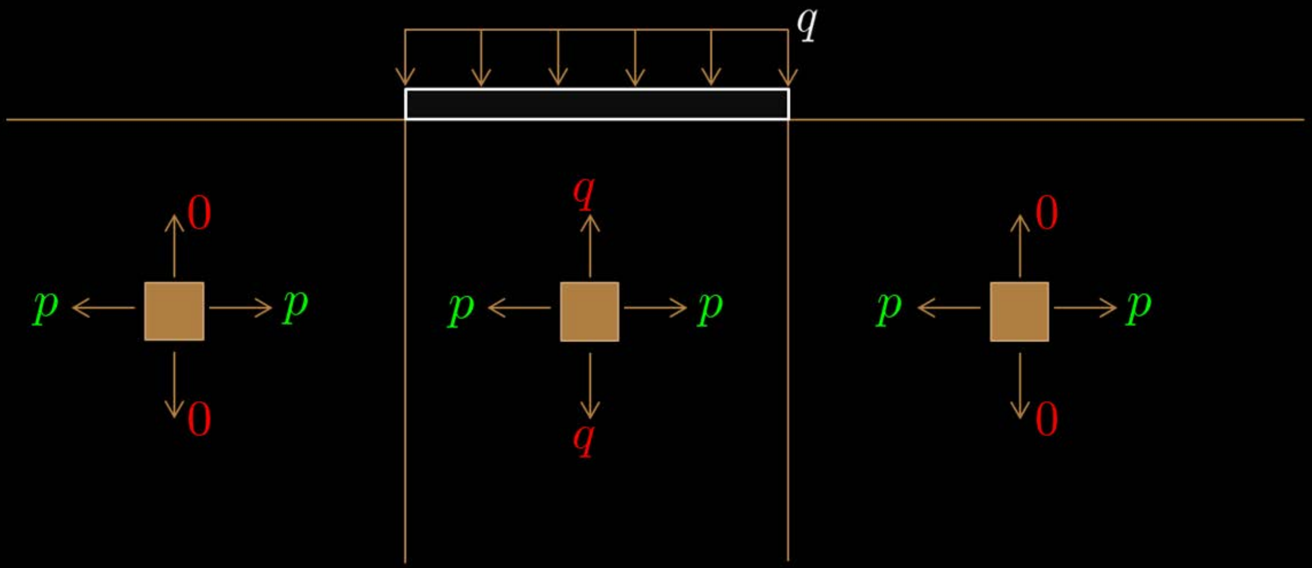
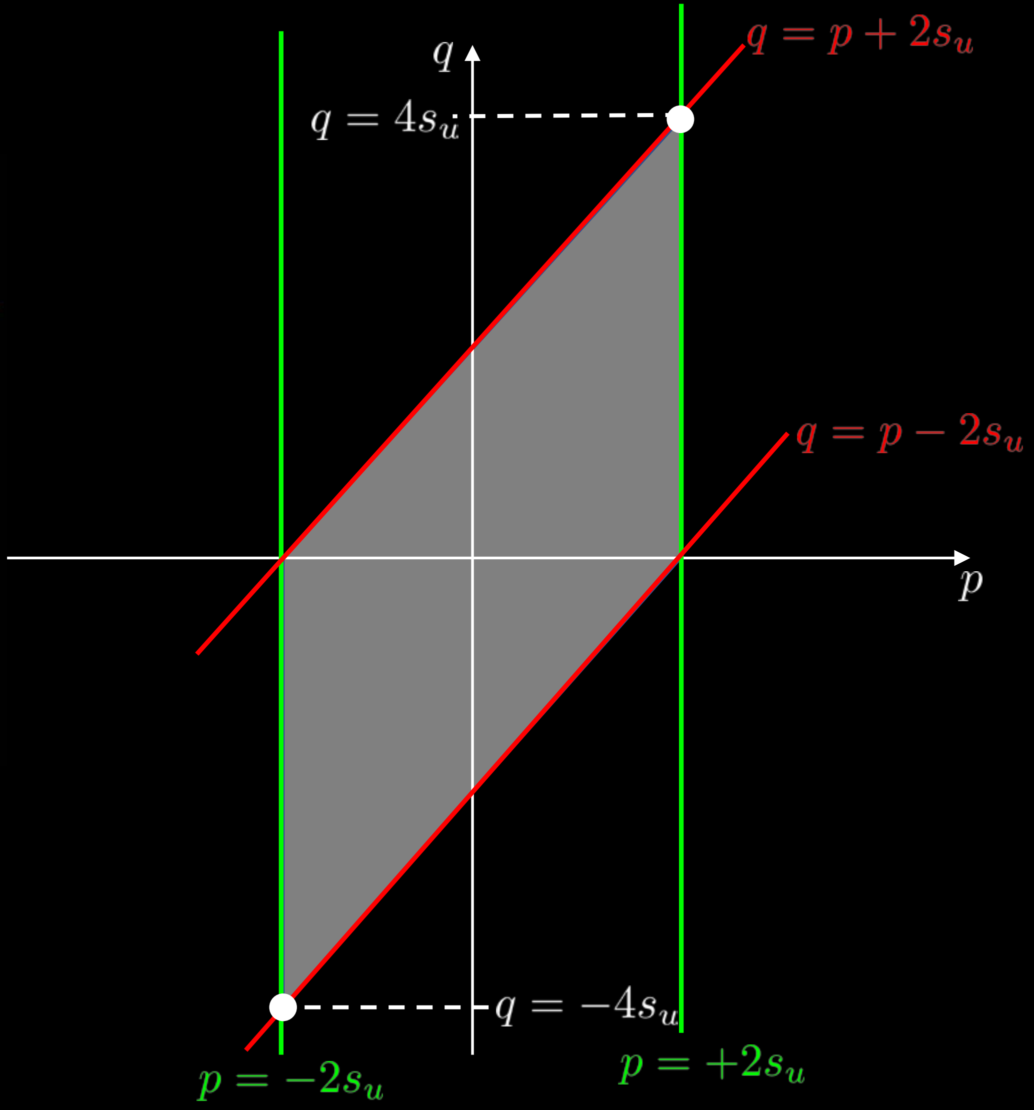


The Optum approach

Example: footing on weightless Tresca soil





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The Optum approach

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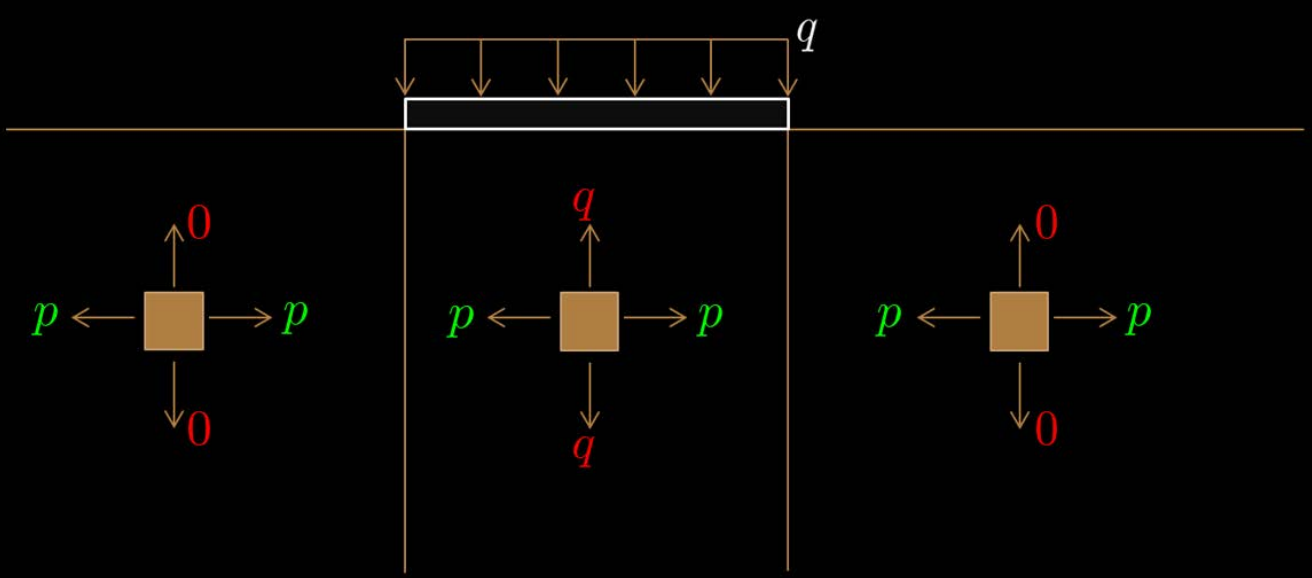

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subject to $|p| \leq 2s_u$

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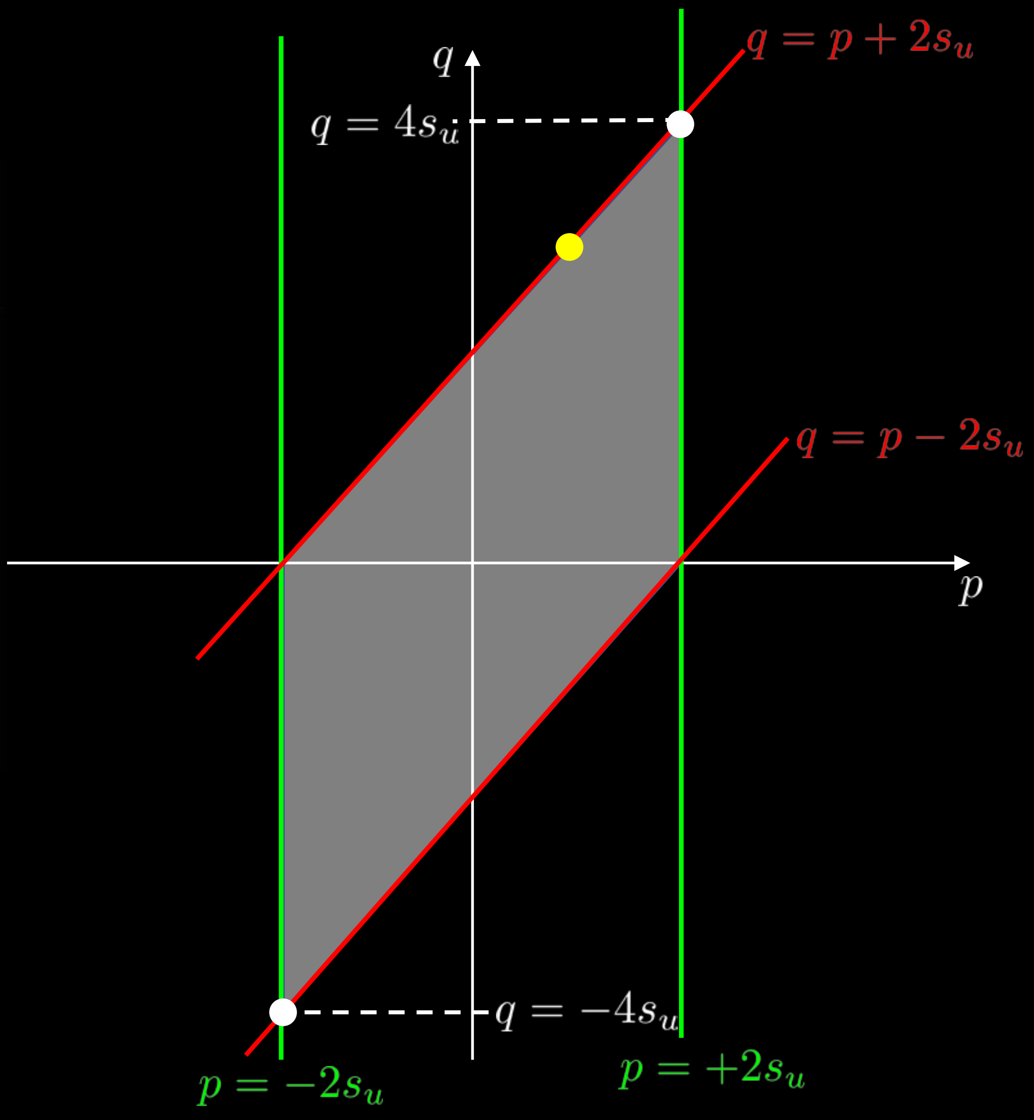

The Optum approach

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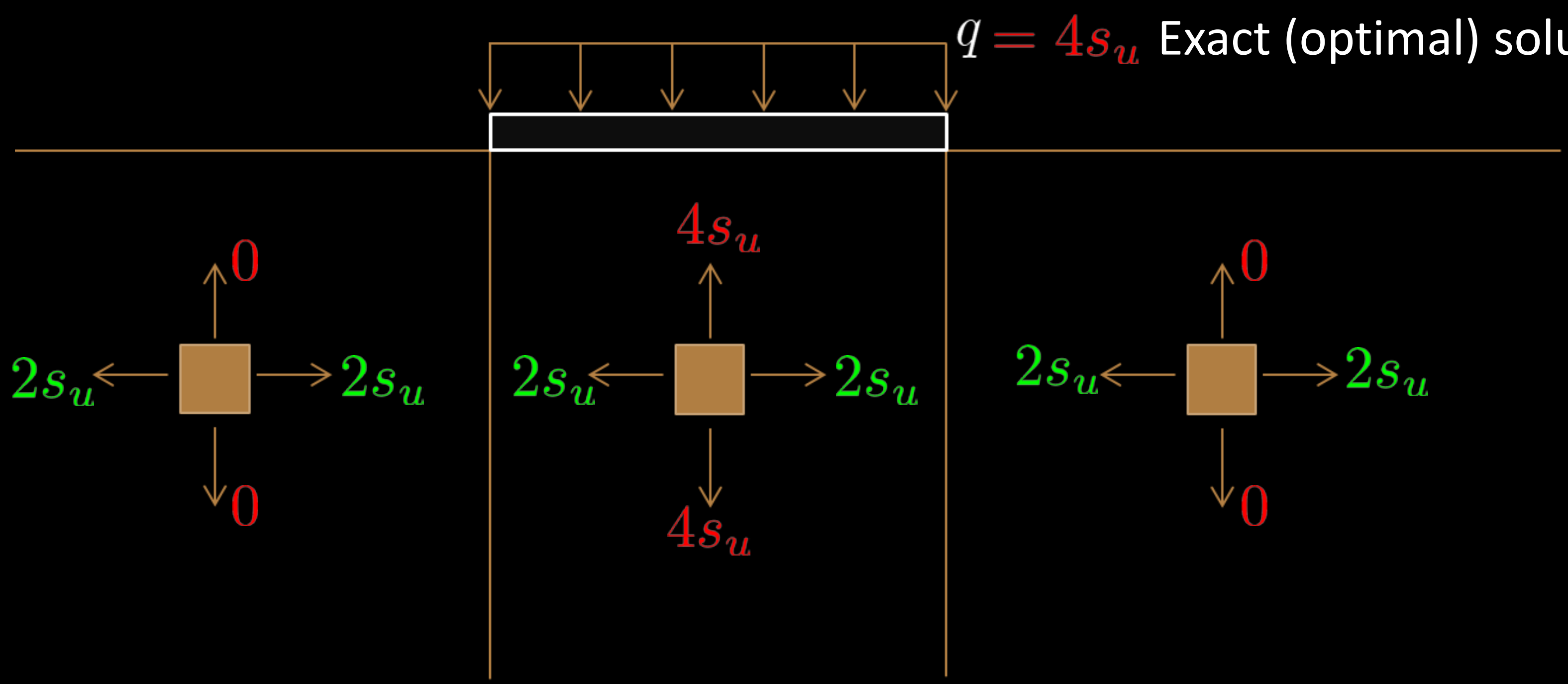

Exact (optimal) solution: $q = 5.14s_u$

maximize q

subject to Global equilibrium (satisfied)

Static BCs

Interface equilibrium

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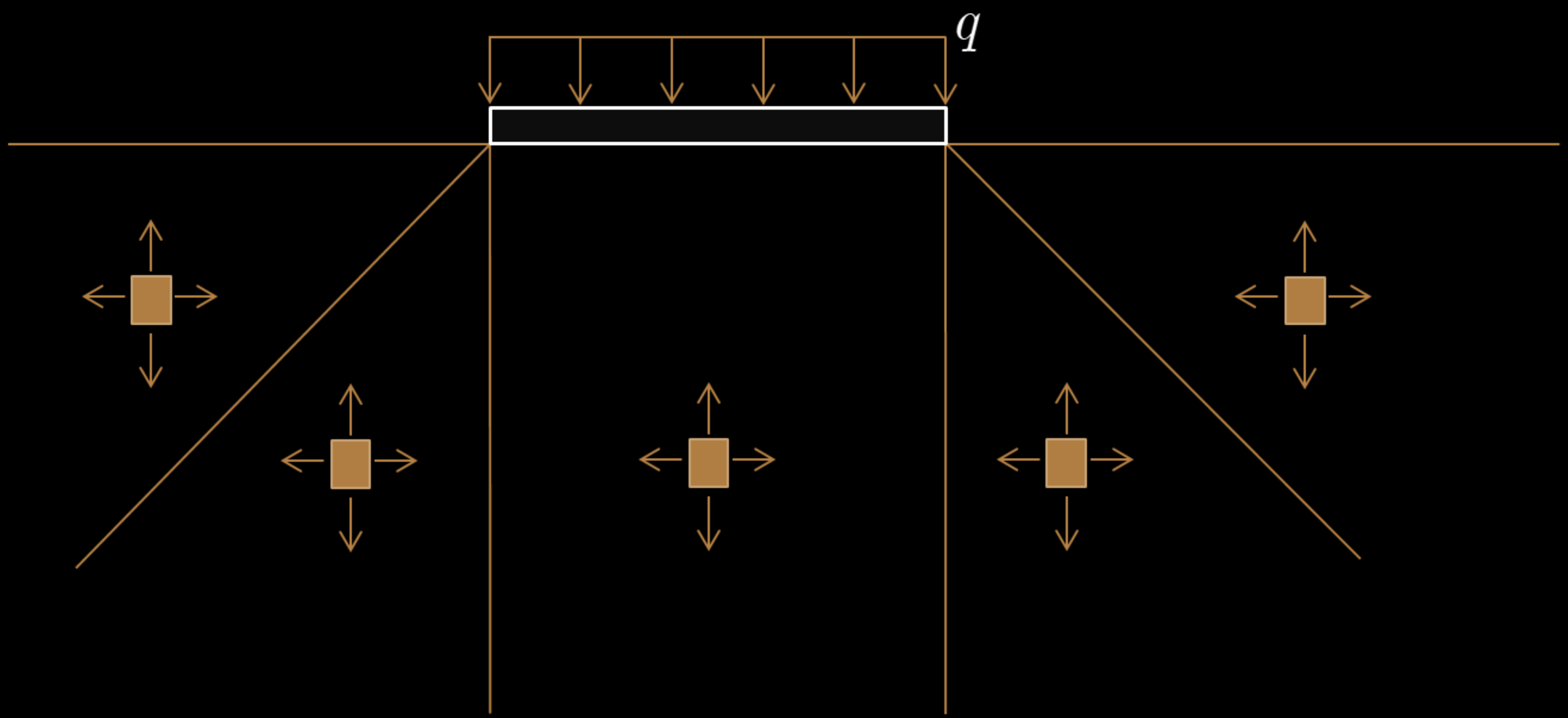
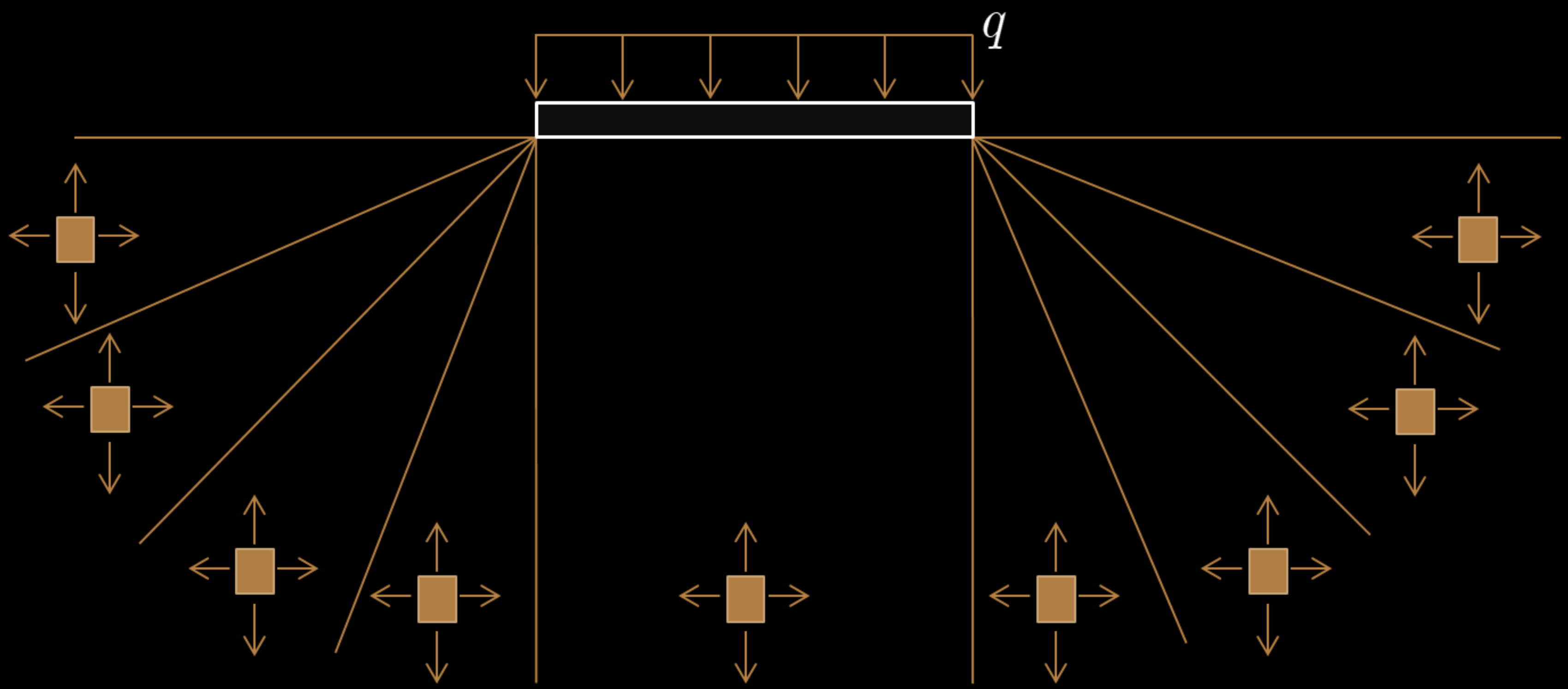
The Optum approach

Example: footing on weightless Tresca soil



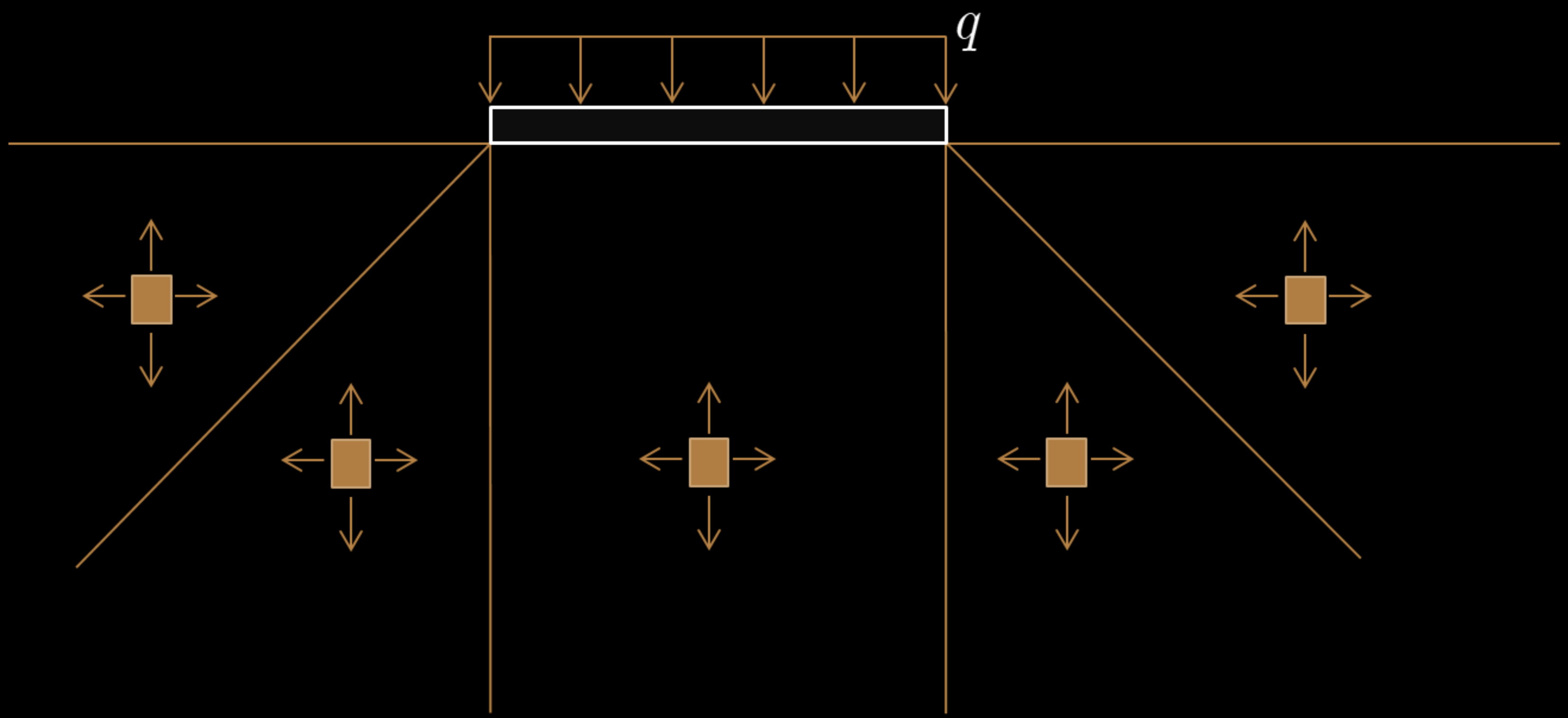
The Optum approach

Example: footing on weightless Tresca soil



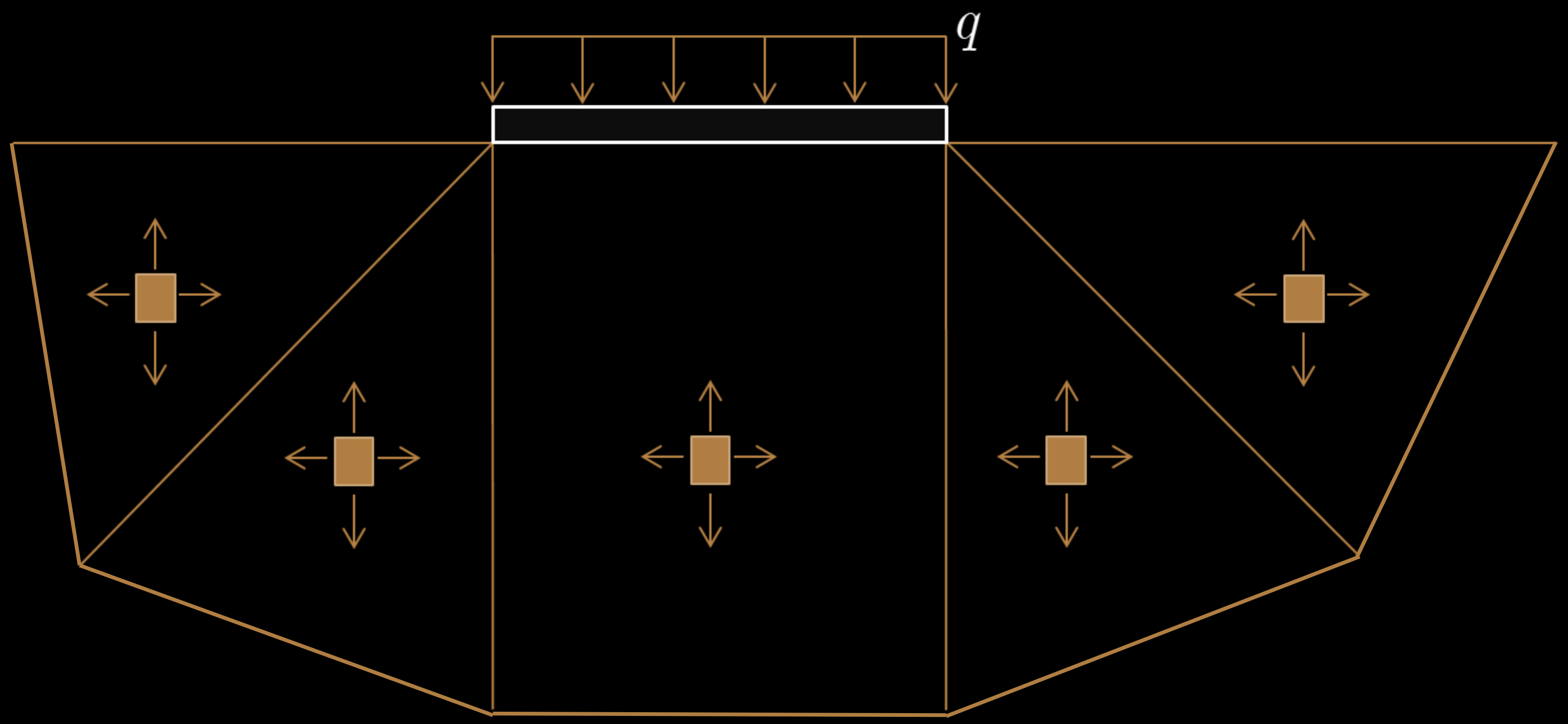
The Optum approach

Example: footing on weightless Tresca soil



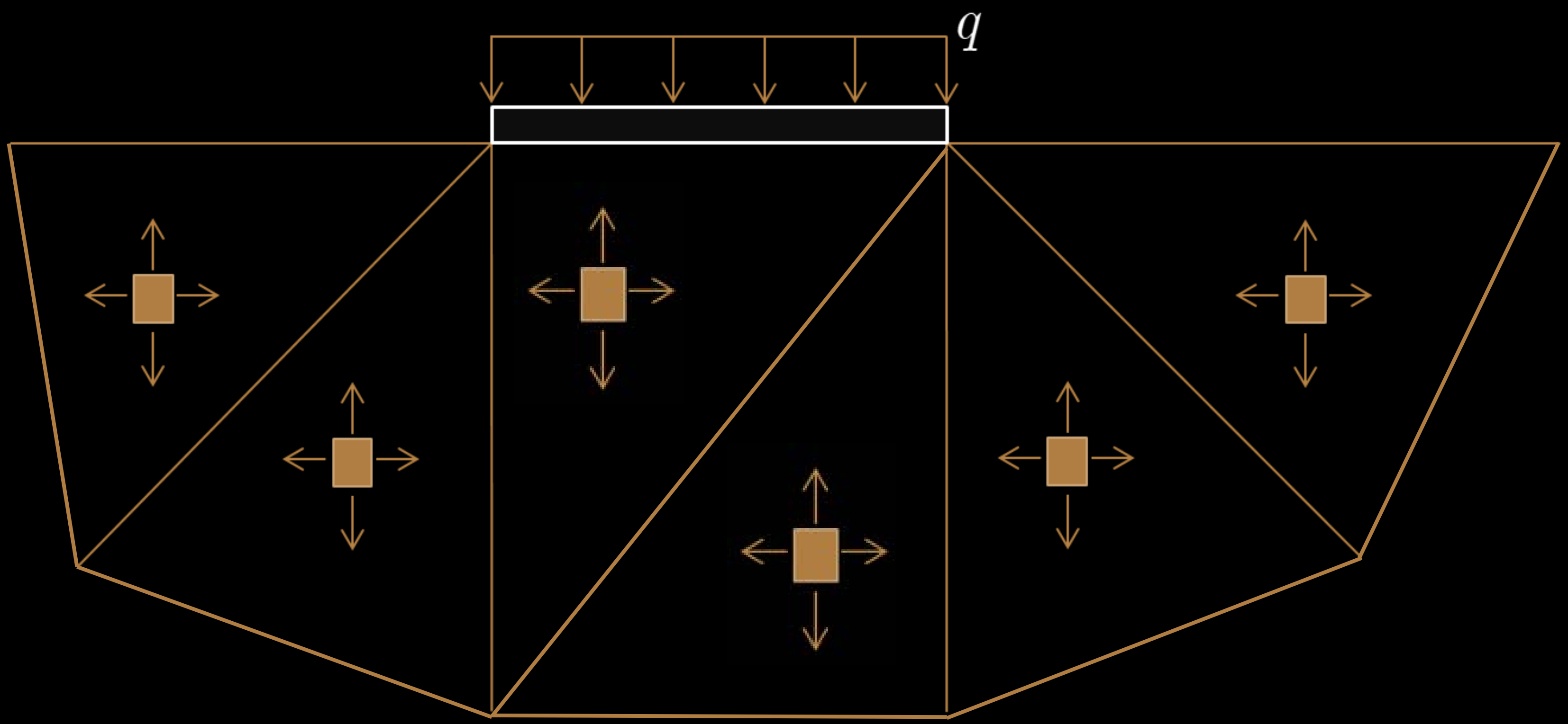
The Optum approach

Example: footing on weightless Tresca soil



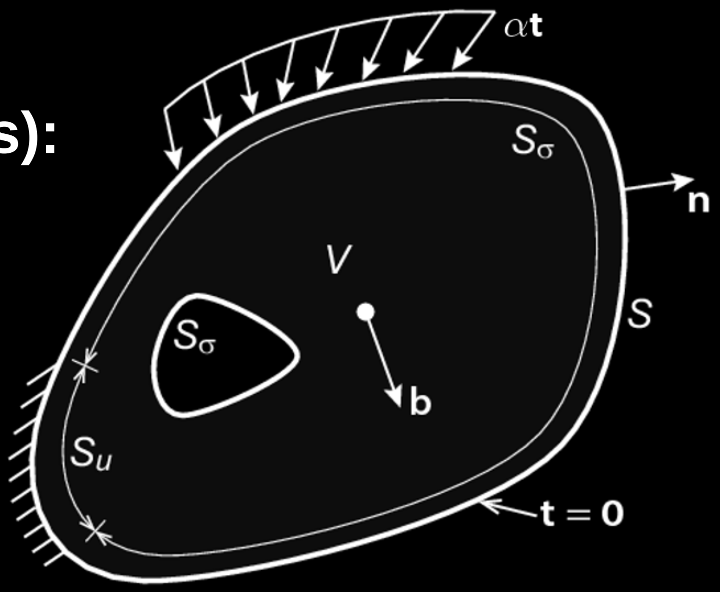
The Optum approach

Example: footing on weightless Tresca soil



The Optum approach

Variational principle (equivalent to governing equations):



$$\begin{aligned} \text{maximize} \quad & \alpha \\ \text{subject to} \quad & \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \mathbf{0} \\ & \boldsymbol{n} \cdot \boldsymbol{\sigma} = \alpha \boldsymbol{t} \ \text{ on } S_\sigma \\ & F(\boldsymbol{\sigma}) \leq 0 \end{aligned}$$

Discretize: replace continuous variables with discrete counterparts to obtain discrete optimization problem:

$$\begin{aligned} \text{maximize} \quad & \alpha \\ \text{subject to} \quad & \boldsymbol{B}^\mathsf{T} \boldsymbol{\sigma} = \alpha \boldsymbol{f} + \boldsymbol{f}_\mathbf{0} \\ & F(\boldsymbol{\sigma}) \leq 0 \end{aligned}$$

The Optum approach

Variational principle (equivalent to governing equations):

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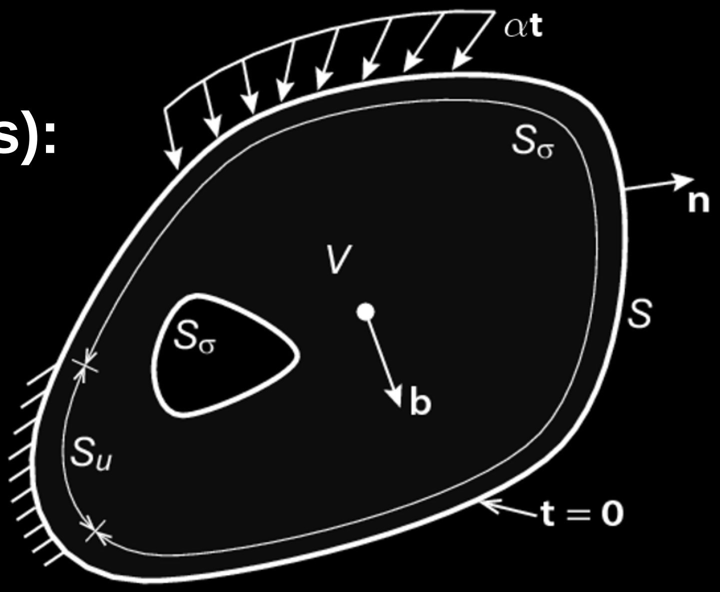

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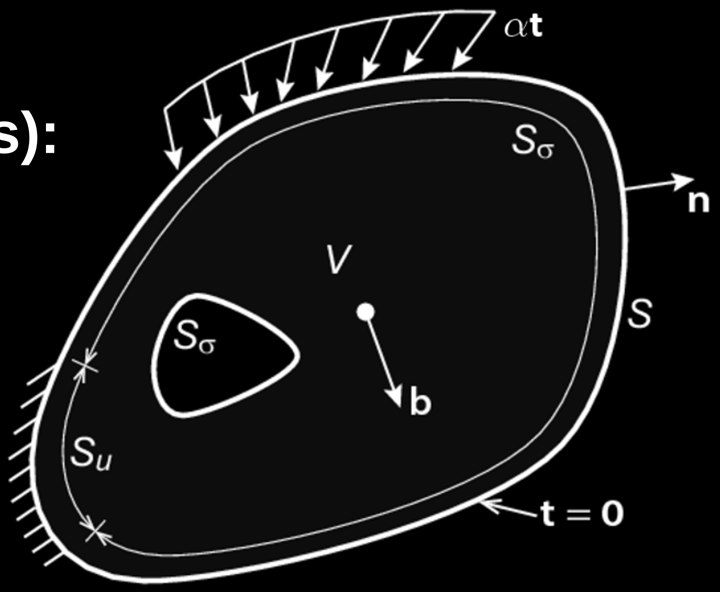
Very efficient and robust algorithms now available (conic programming)

The Optum approach

Variational principle (equivalent to governing equations):



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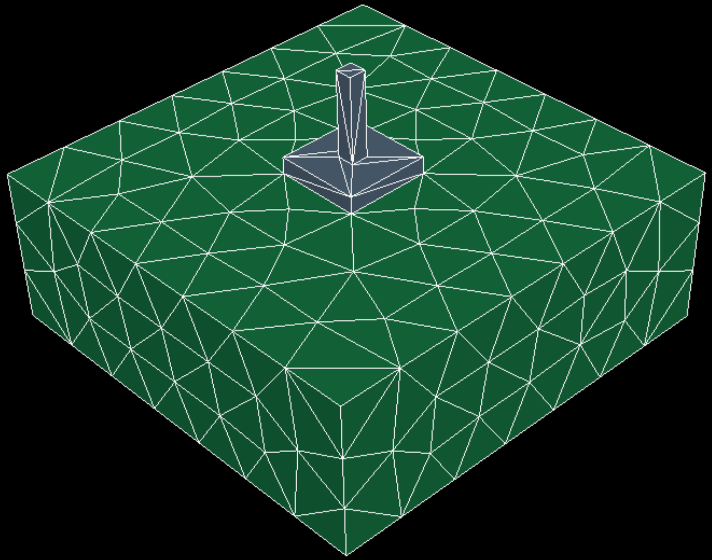
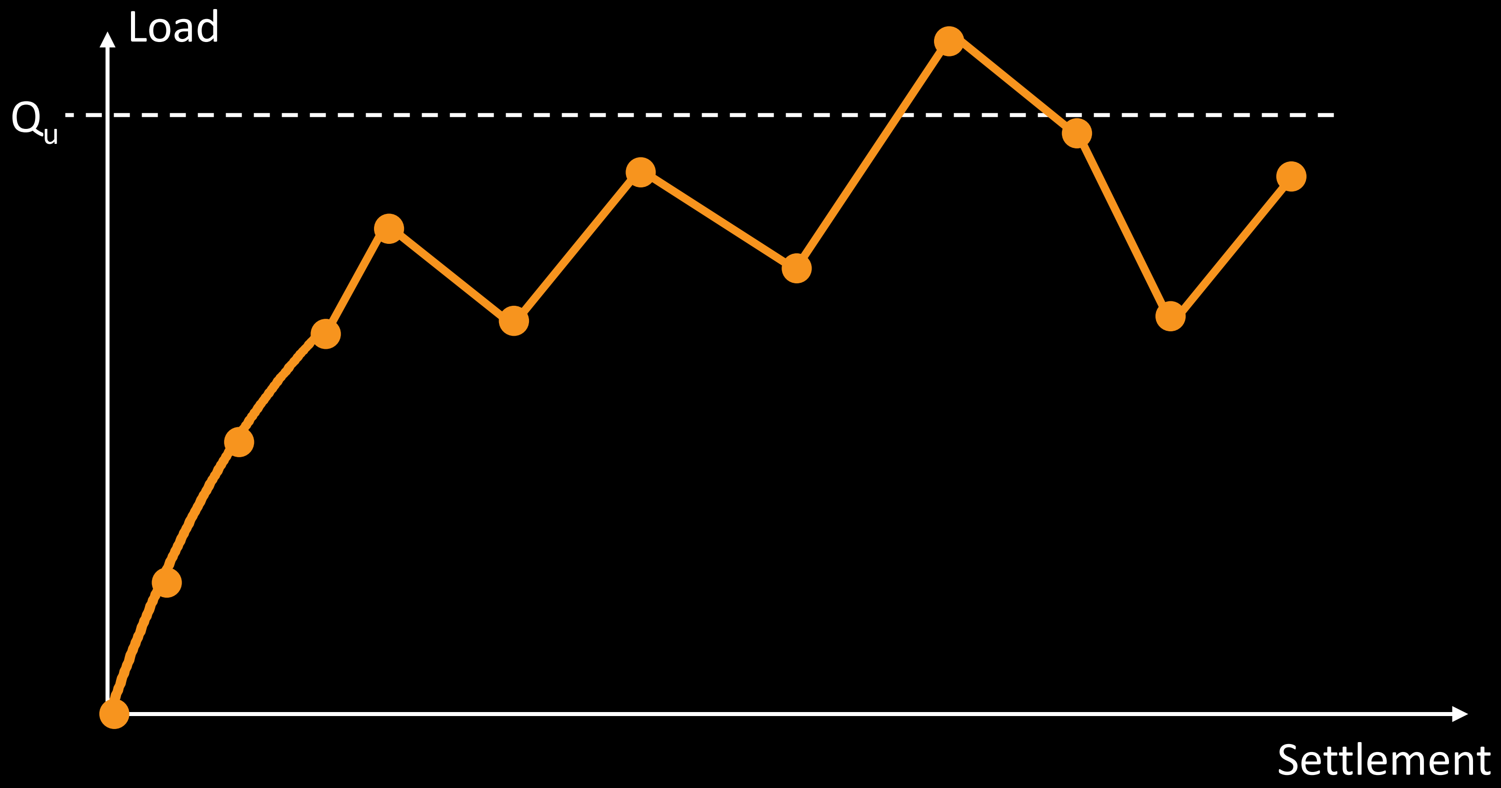
Upper bound formulation takes same form but with slightly different B, f, f_0

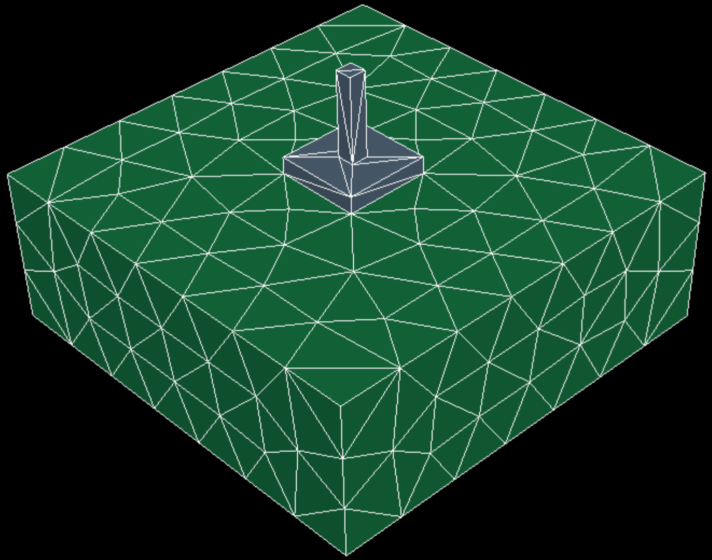
The conventional FE approach



Increase load to failure:



The Optum approach



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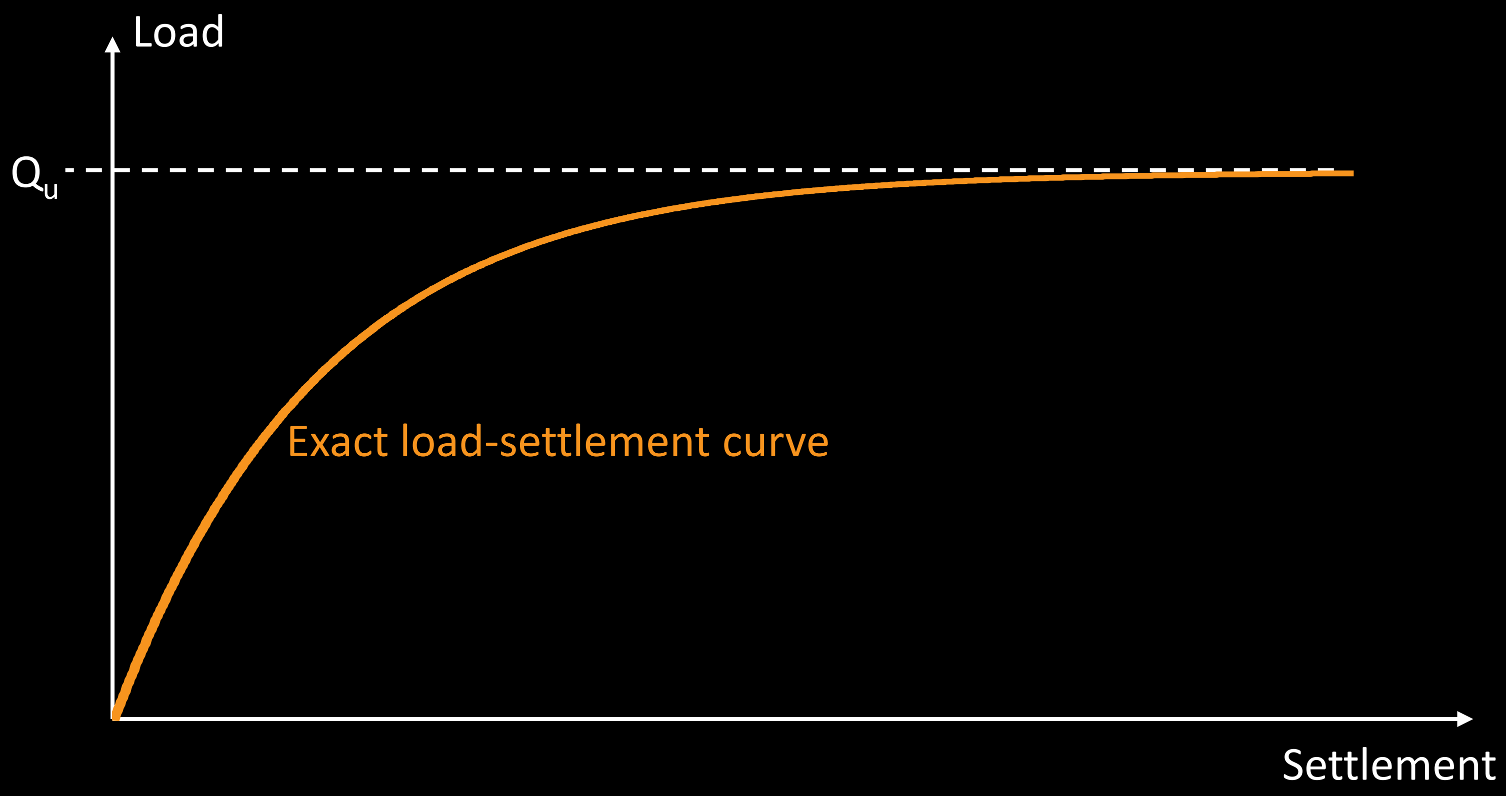
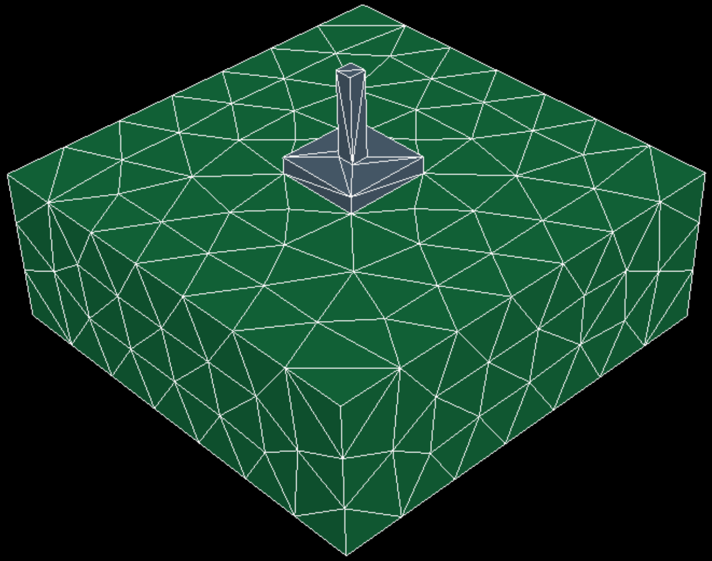
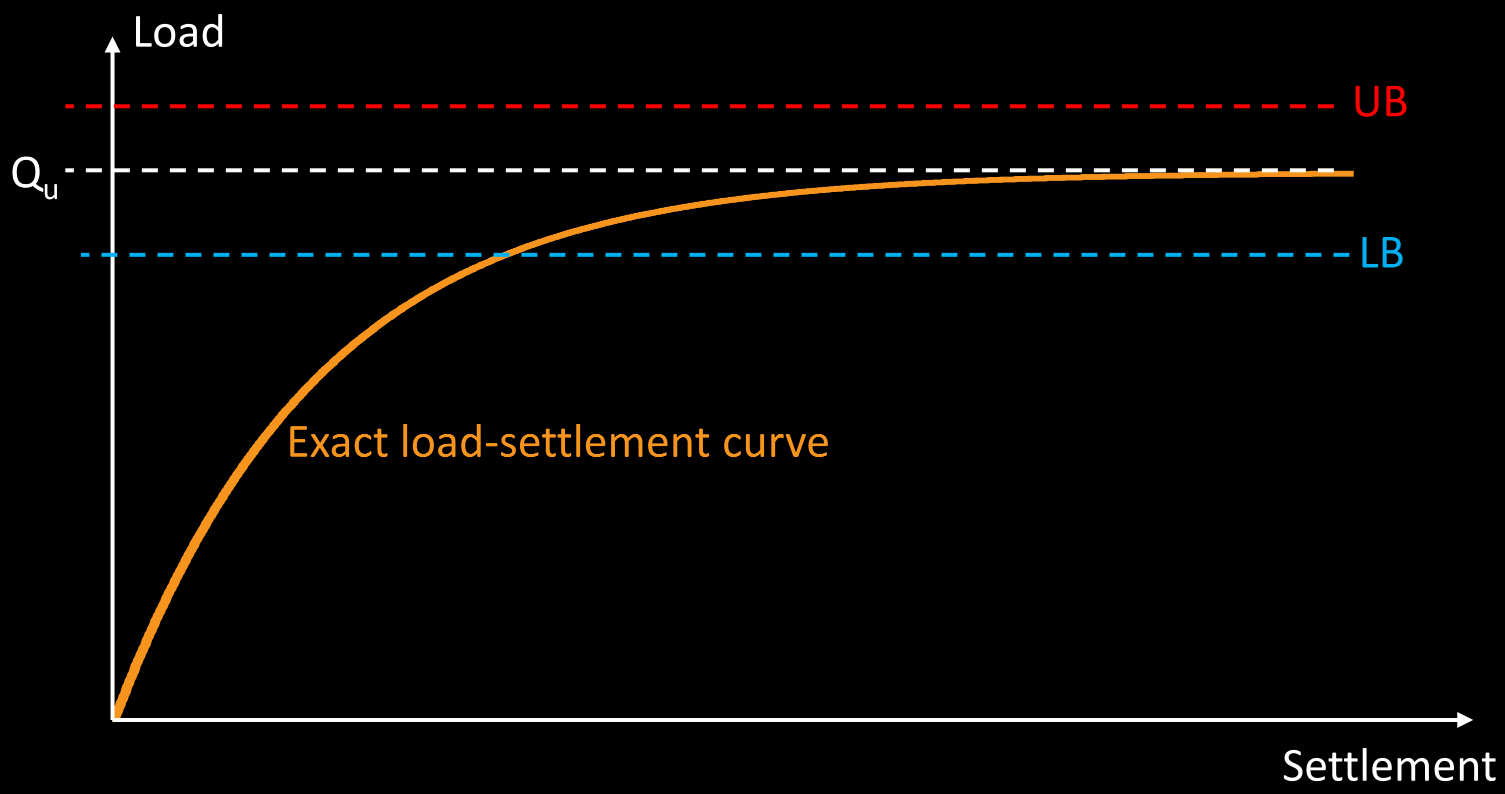


What is the limit load?

The Optum approach

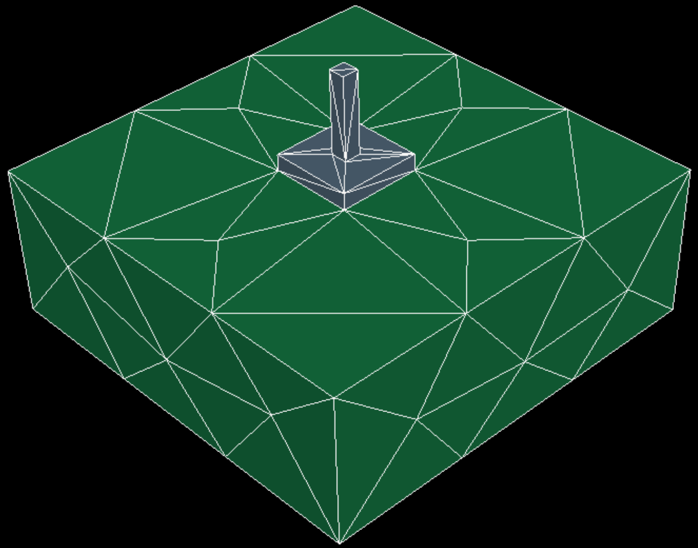


Increase load to failure:

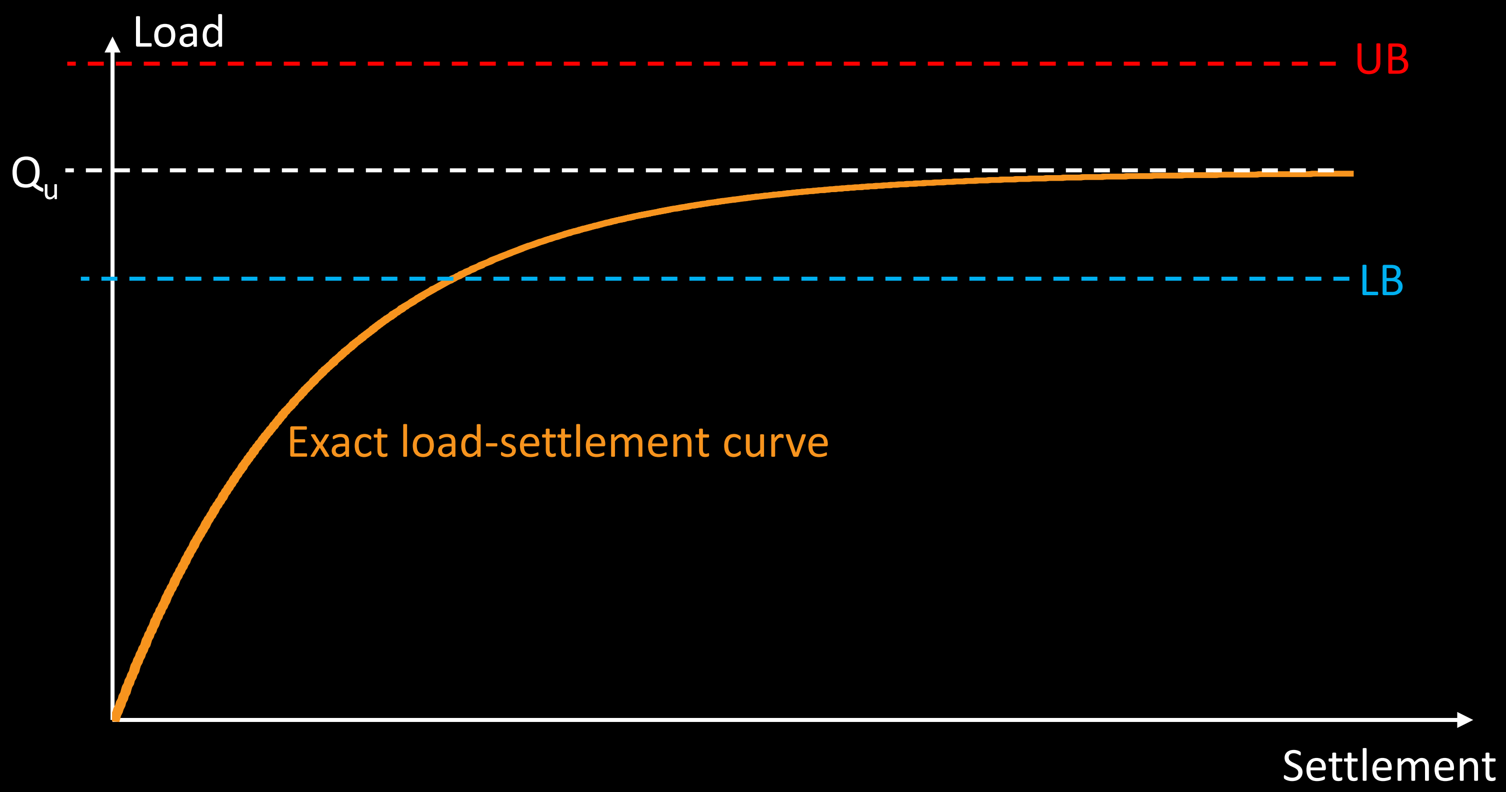


What is the limit load?

The Optum approach

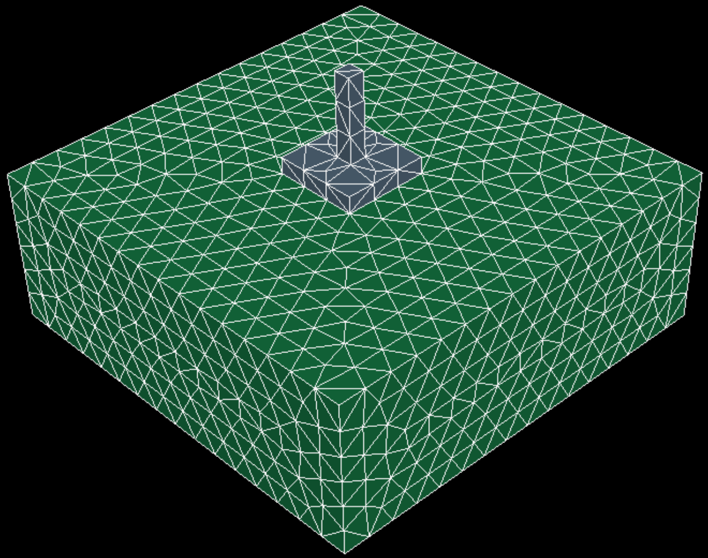


Increase load to failure:

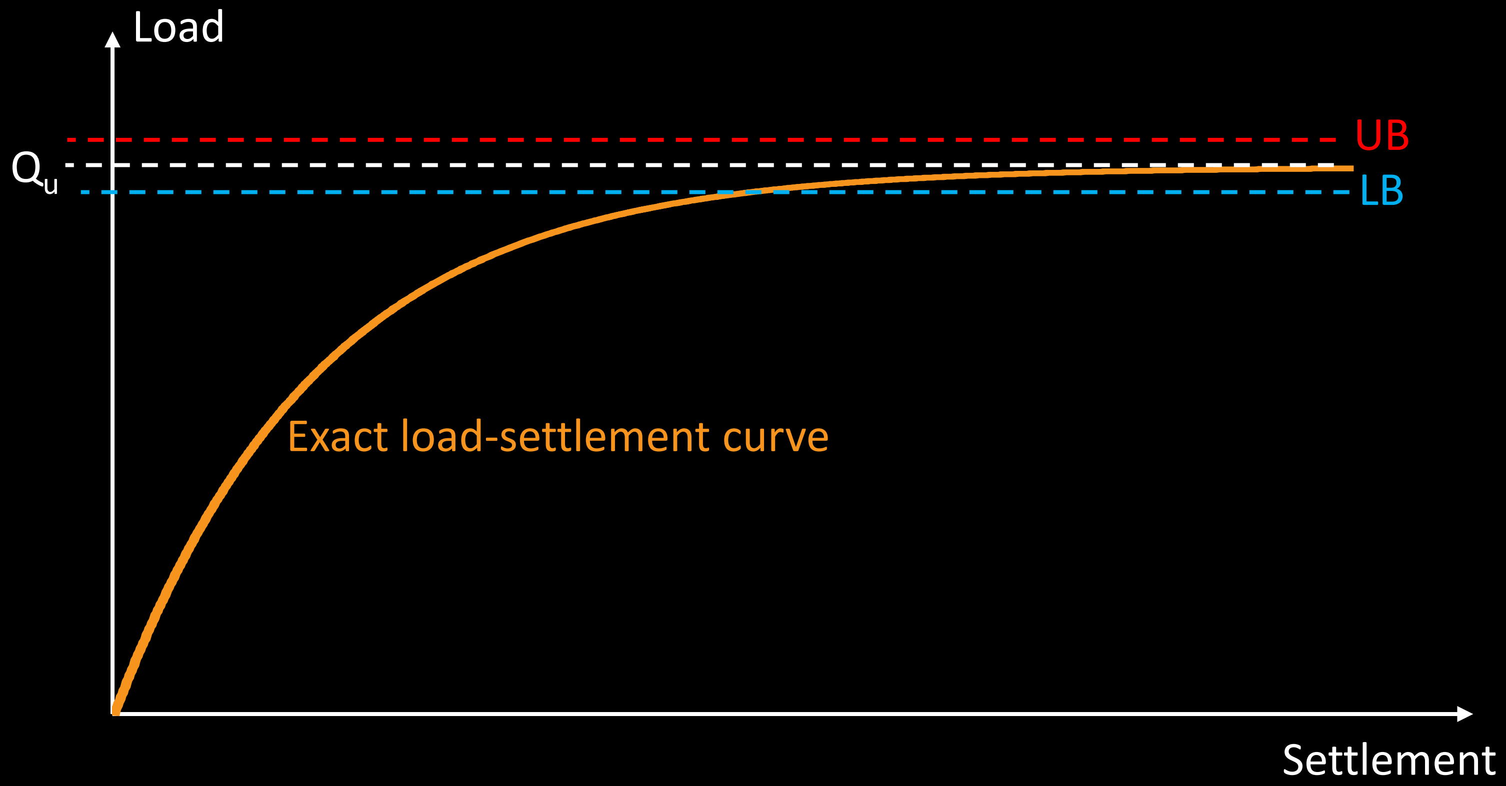


How accurate is the solution?

The Optum approach

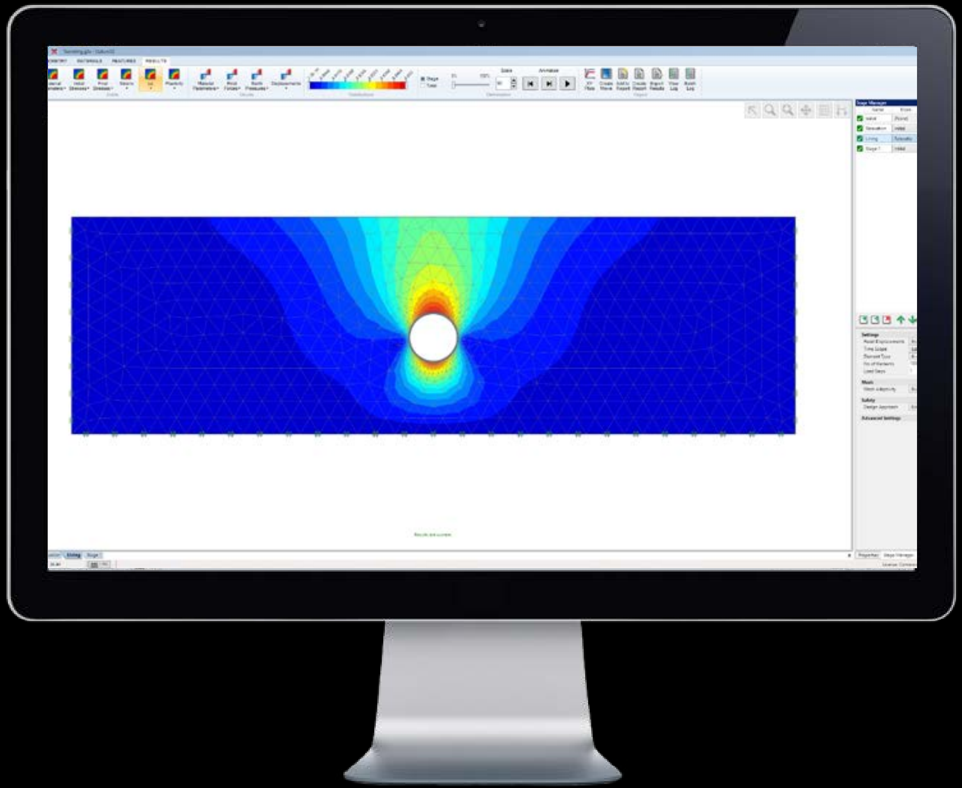


Increase load to failure:

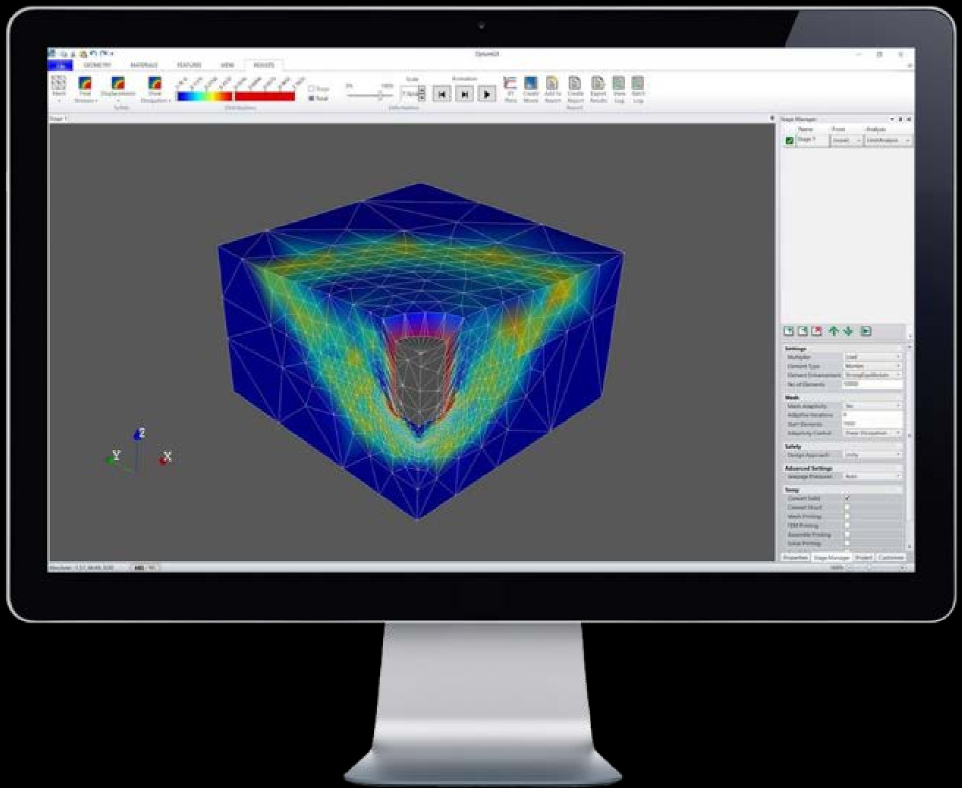


How accurate is the solution?

Examples



OPTUM G2

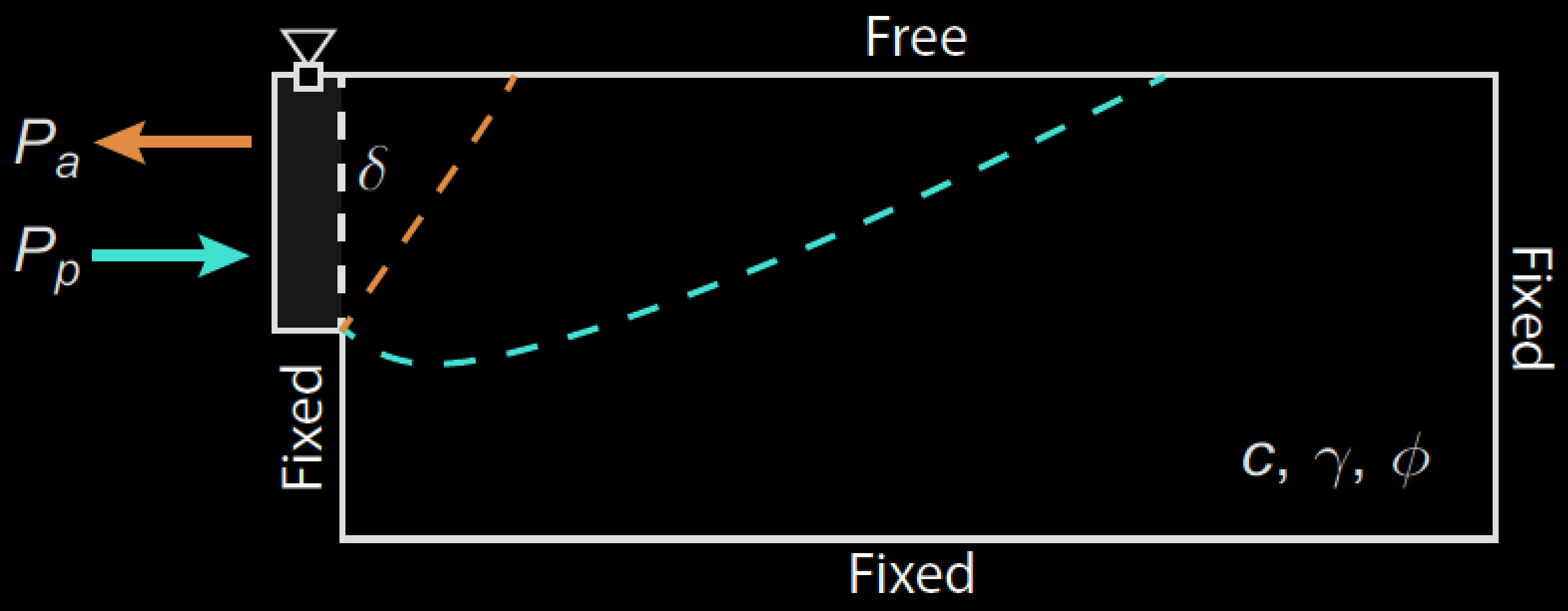


OPTUM G3

Abirtrary problems can now be solved – to within a verified accuracy – in a reasonable amount of time

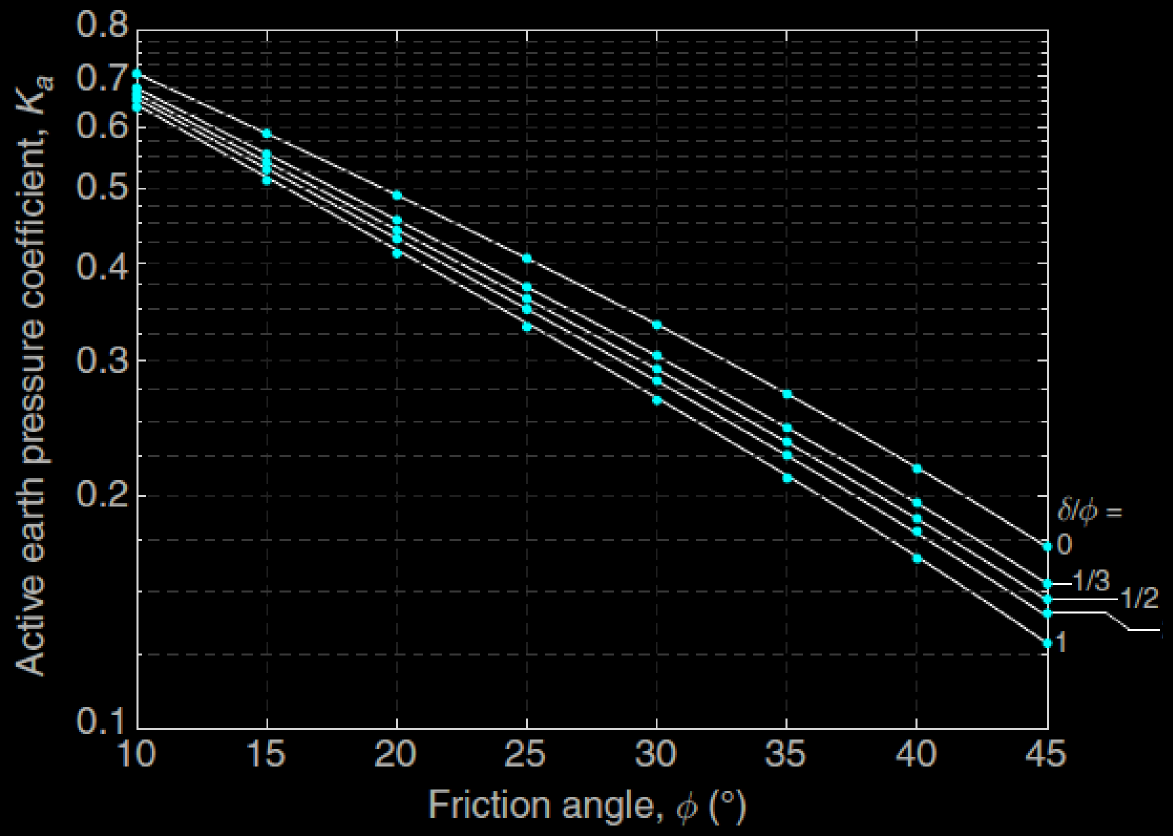
Example

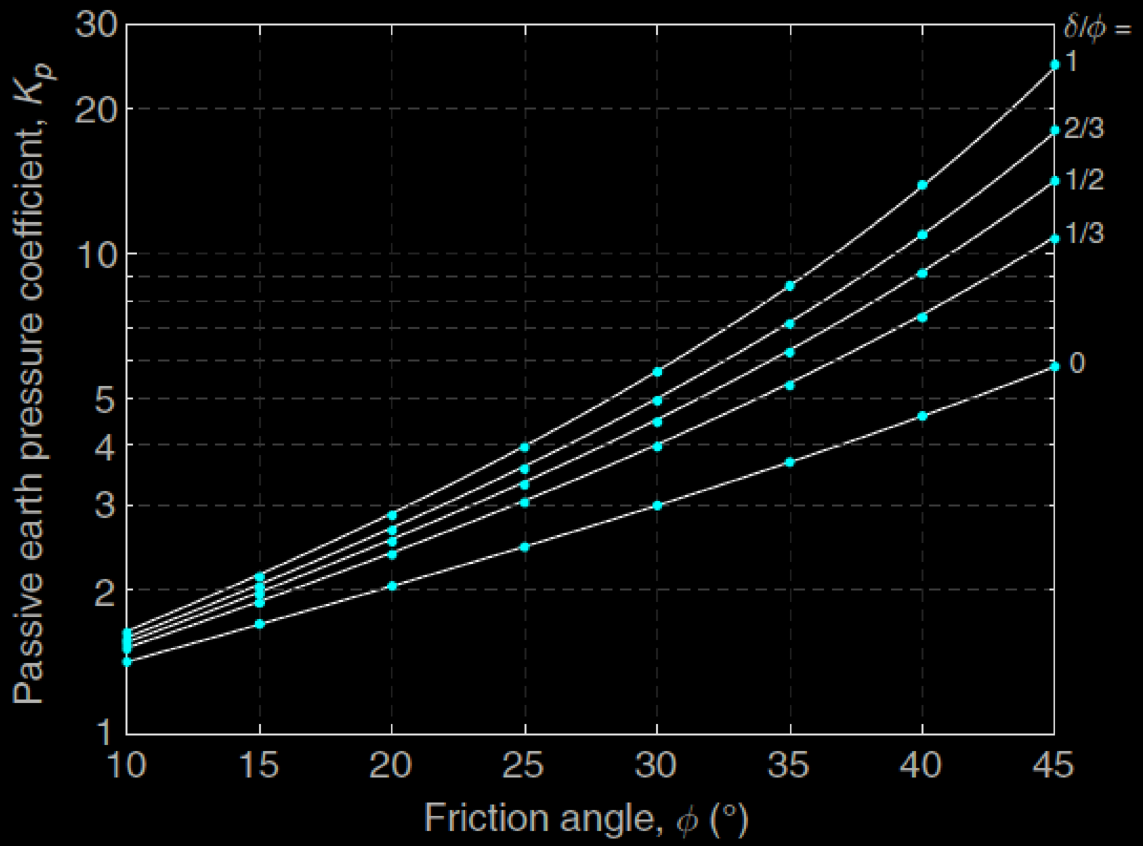
Earth pressure coefficients



Example

Earth pressure coefficients

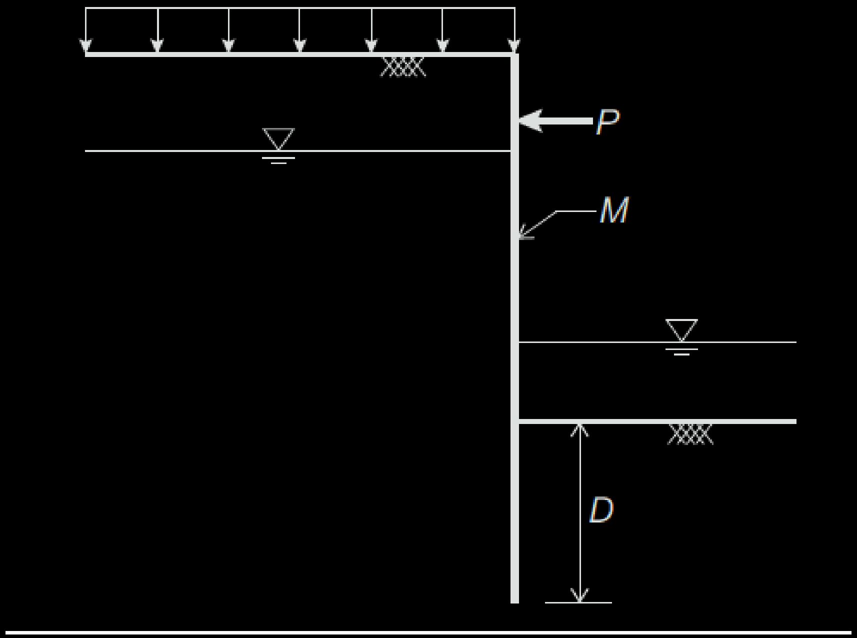




Dots: FELA, curves: fits (see paper). Accuracy = ± 1%

Example

Embedded retaining walls

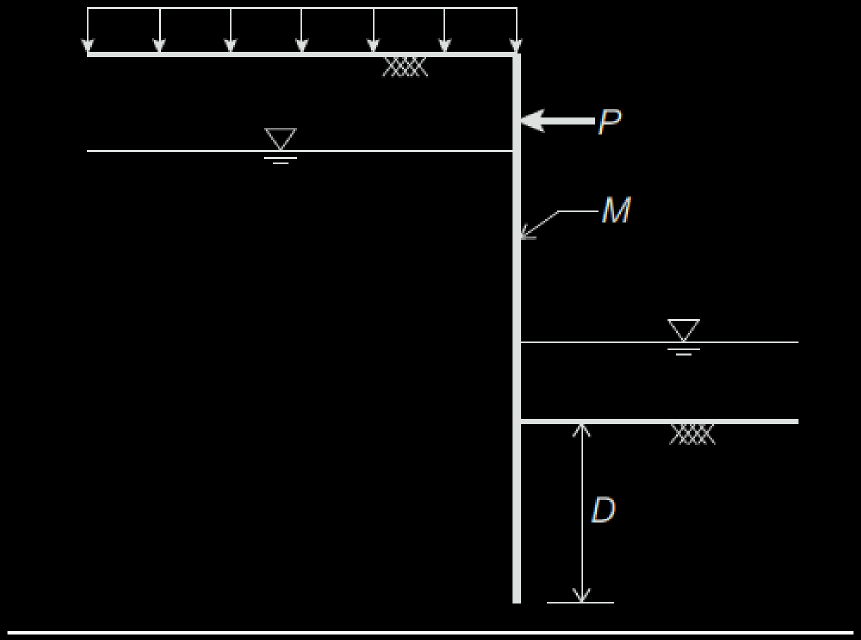


Design variables:

- Wall moment capacity (M)
- Anchor capacity (P)
- Embedment depth (D)

Example

Embedded retaining walls



Design variables:

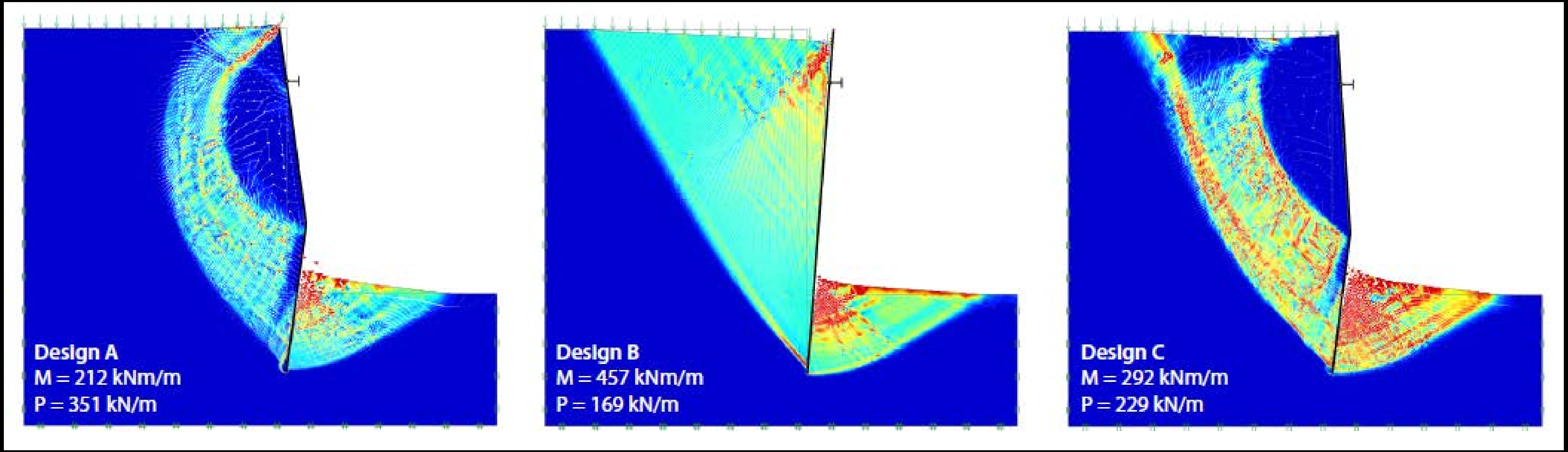
+ Wall moment capacity (M)
+ Anchor capacity (P)
+ Embedment depth (D)

For given D:

+ Minimize wall moment
+ Minimize anchor capacity
+ Minimize combination of wall moment capacity and anchor capacity

Example

Embedded retaining walls



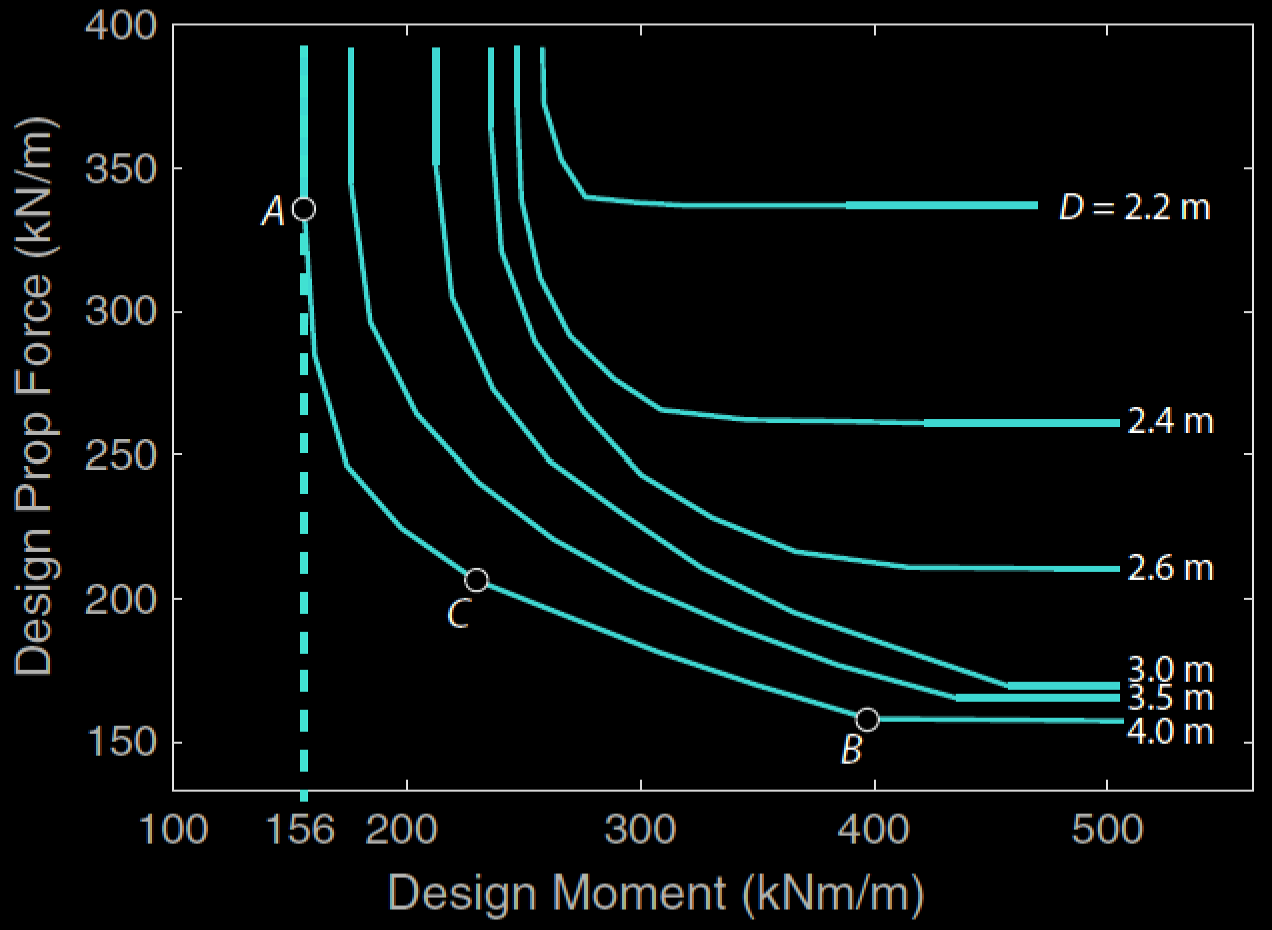
Min M (large P)

Min P (large M)

Compromise

Example

Embedded retaining walls



The Optum approach

Rigid plasticity

Equilibrium: $\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0}$

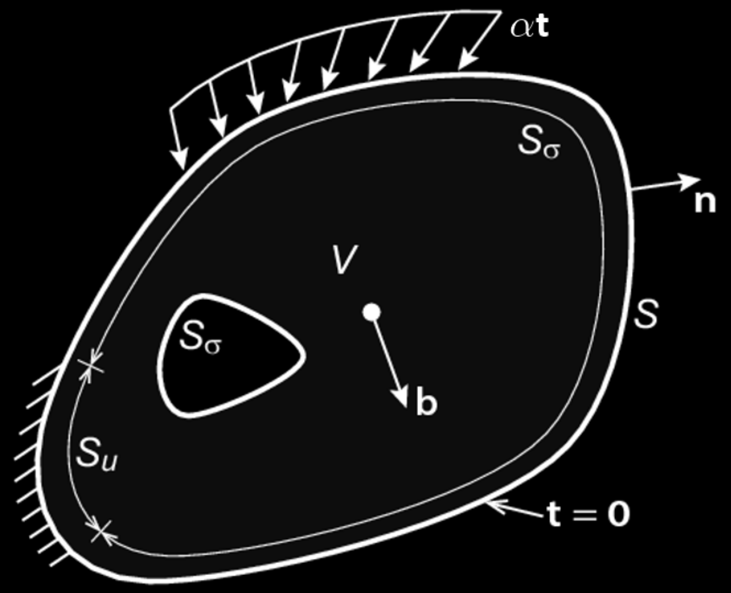
Static BC: $\boldsymbol{n} \cdot \boldsymbol{\sigma} = \alpha \boldsymbol{t}$ on S_σ

Strain-disp: $\dot{\boldsymbol{\varepsilon}}^p = \nabla \dot{\boldsymbol{u}}$

Yield condition: $F(\boldsymbol{\sigma}) \leq 0$

Flow rule: $\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \dfrac{\partial F}{\partial \boldsymbol{\sigma}}$

Complementarity: $\dot{\lambda} \geq 0, \quad \dot{\lambda} F(\boldsymbol{\sigma}) = 0$



Variational formulation

maximize α

subject to $\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0}$

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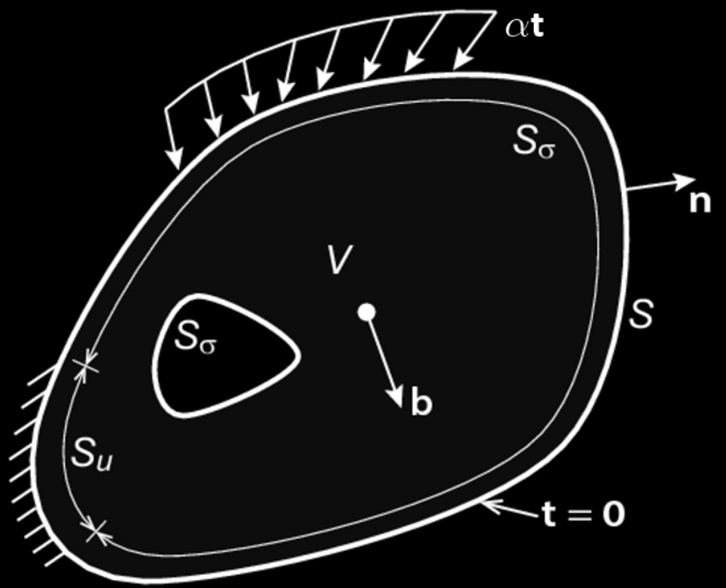

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What about settlements (elastoplasticity)?

The Optum approach

Elastoplasticity

Equilibrium: $\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0}$

Static BC: $\boldsymbol{n} \cdot \boldsymbol{\sigma} = \alpha \boldsymbol{t}$ on S_σ

Strain-disp-Hooke: $\nabla \dot{\boldsymbol{u}} = \mathbb{C}\dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{\varepsilon}}^p$

Yield condition: $F(\boldsymbol{\sigma}) \leq 0$

Flow rule: $\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \dfrac{\partial F}{\partial \boldsymbol{\sigma}}$

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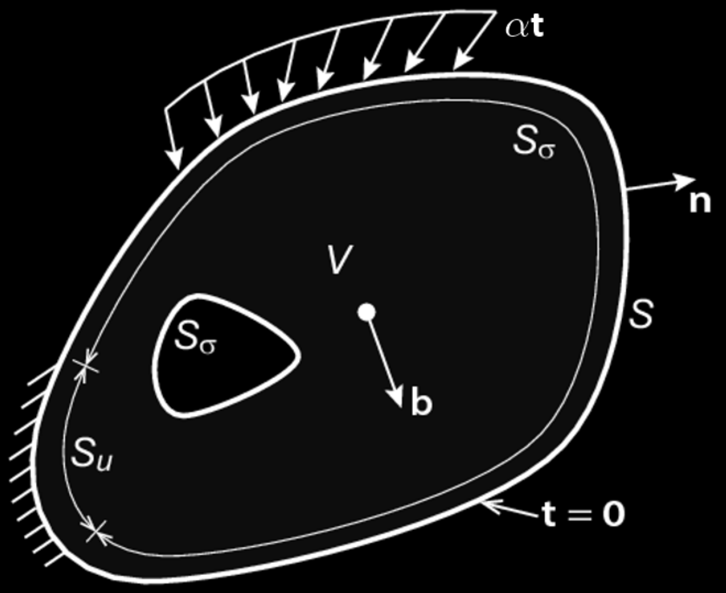

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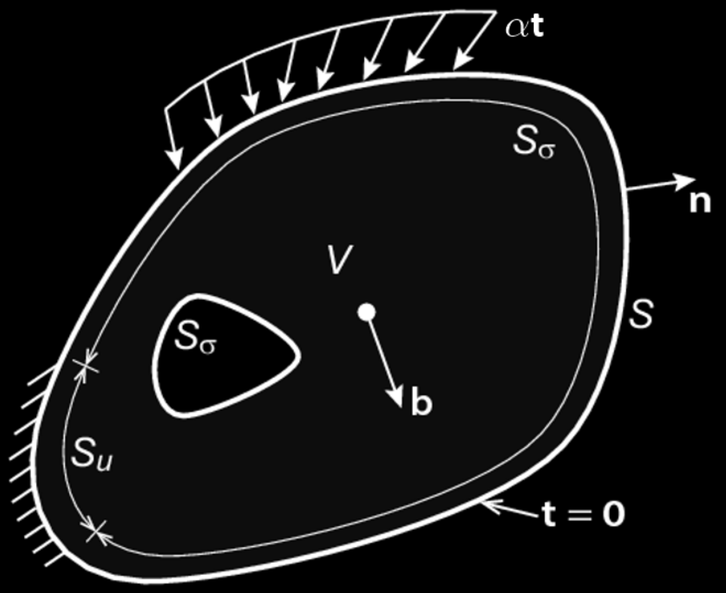


Variational formulation

maximize $\alpha - \int_V \frac{1}{2} \Delta\boldsymbol{\sigma} \cdot \mathbb{C} \Delta\boldsymbol{\sigma} \, \mathrm{d}V$

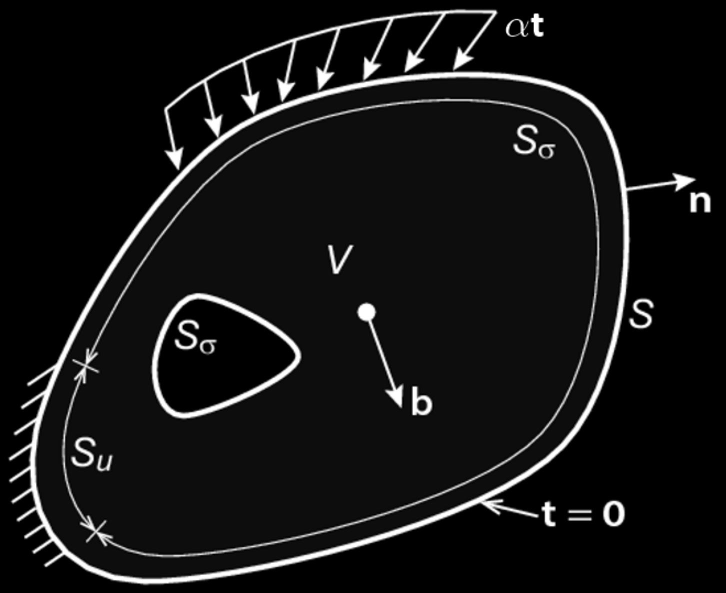
subject to $\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \mathbf{0}$

$\boldsymbol{n} \cdot \boldsymbol{\sigma} = \alpha \boldsymbol{t}$ on S_σ

$F(\boldsymbol{\sigma}) \leq 0$

What about settlements (elastoplasticity)?

The Optum approach



Rigid plasticity

maximize $\quad \alpha$

subject to $\quad \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0}$

$\boldsymbol{n} \cdot \boldsymbol{\sigma} = \alpha \boldsymbol{t}$ on S_σ

$F(\boldsymbol{\sigma}) \leq 0$

Elastoplasticity

maximize $\quad \alpha - \int_V \frac{1}{2} \Delta\boldsymbol{\sigma} \cdot \mathbb{C} \Delta\boldsymbol{\sigma} \, \mathrm{d}V$

subject to $\quad \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0}$

$\boldsymbol{n} \cdot \boldsymbol{\sigma} = \alpha \boldsymbol{t}$ on S_σ

$F(\boldsymbol{\sigma}) \leq 0$

Similar modifications to reproduce governing equations of hardening plasticity, coupled pore pressure-deformation, dynamics, and more

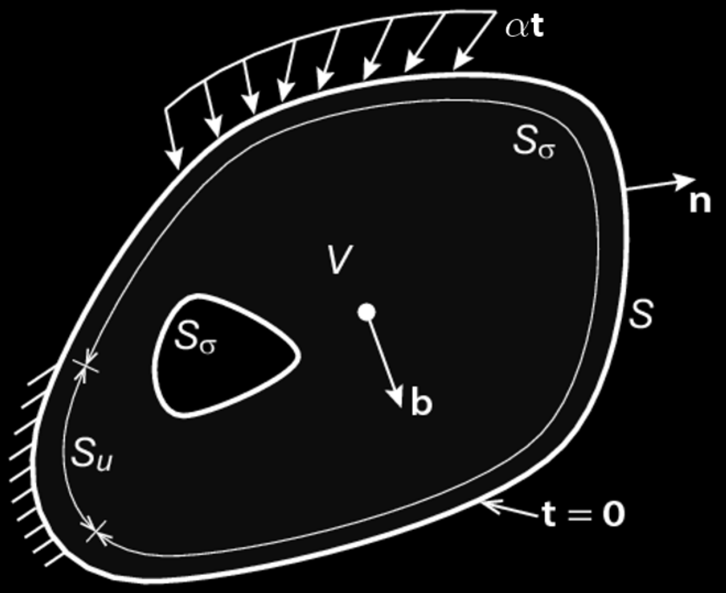
The Optum approach

Rigid plasticity

$$\begin{aligned} &\text{maximize} && \alpha \\ &\text{subject to} && \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \mathbf{0} \\ &&& \boldsymbol{n} \cdot \boldsymbol{\sigma} = \alpha \boldsymbol{t} \quad \text{on } S_\sigma \\ &&& F(\boldsymbol{\sigma}) \leq 0 \end{aligned}$$

Elastoplasticity

$$\begin{aligned} &\text{maximize} && \alpha - \int_V \tfrac{1}{2} \Delta\boldsymbol{\sigma} \cdot \mathbb{C} \Delta\boldsymbol{\sigma} \, \mathrm{d}V \\ &\text{subject to} && \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \mathbf{0} \\ &&& \boldsymbol{n} \cdot \boldsymbol{\sigma} = \alpha \boldsymbol{t} \quad \text{on } S_\sigma \\ &&& F(\boldsymbol{\sigma}) \leq 0 \end{aligned}$$



OPTUM is general FE – not just limit analysis – but approached from the view of Euler's maximum/minimum postulate

The Optum approach

Alernative narrative: limit analysis follows as a special (and useful) case of elastoplasticity:

$$\begin{aligned} &\text{maximize} && \alpha - \int_V \tfrac{1}{2}\Delta\boldsymbol{\sigma}\cdot\mathbb{C}\Delta\boldsymbol{\sigma}\,\mathrm{d}V \\ &\text{subject to} && \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}+\boldsymbol{b}=\boldsymbol{0} \\ &&& \boldsymbol{n}\cdot\boldsymbol{\sigma}=\alpha\boldsymbol{t}\ \text{ on } S_\sigma \\ &&& F(\boldsymbol{\sigma})\le 0 \end{aligned}$$

OPTUM is general FE – not just limit analysis – but approached from the view of Euler's maximum/minimum postulate

The Optum approach

Alernative narrative: limit analysis follows as a special (and useful) case of elastoplasticity:

$$
\begin{aligned}
\text{maximize} \quad & \alpha \\
\text{subject to} \quad & \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0} \\
& \boldsymbol{n} \cdot \boldsymbol{\sigma} = \alpha \boldsymbol{t} \quad \text{on } S_\sigma \\
& F(\boldsymbol{\sigma}) \leq 0
\end{aligned}
$$

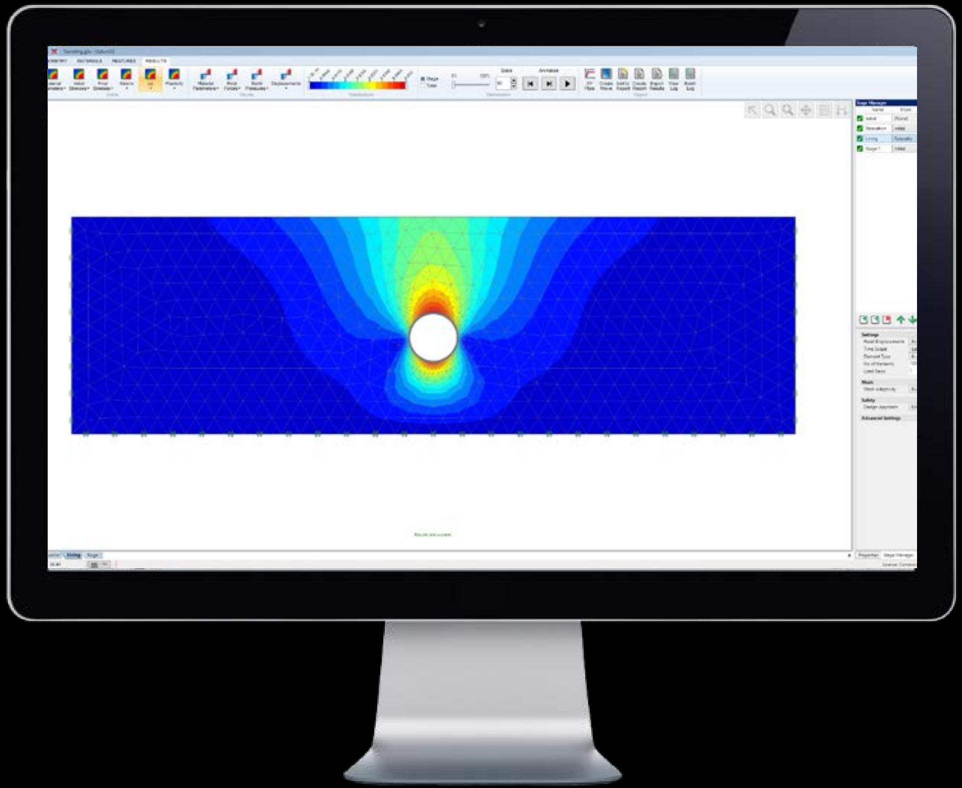
OPTUM is general FE – not just limit analysis – but approached from the view of Euler's maximum/minimum postulate

The Optum approach

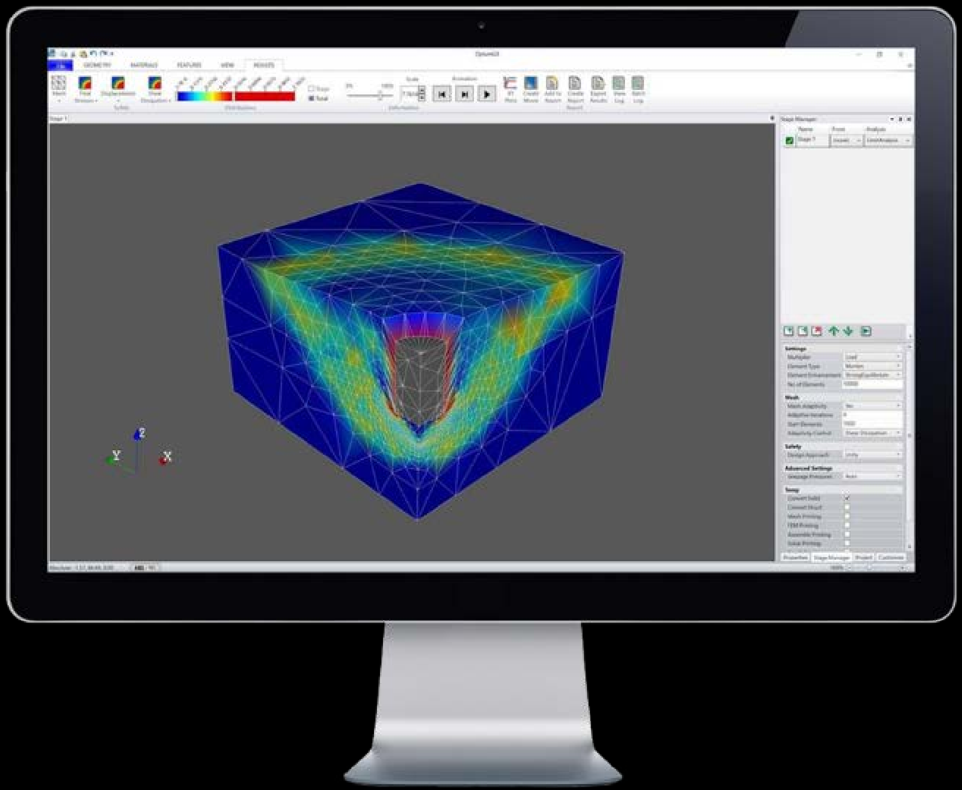
Summary

- Same governing equations but approached differently
- No limitations in terms of physics that can be dealt with
- Significant advantages in terms of computational efficiency and robustness
- Ability to compute rigorous error estimates (upper/lower)
- Ability to provide direct answers to direct questions, e.g. what is the limit load?
- Ability to deal directly with engineering design, c.f. sheet pile design charts

Future developments



OPTUM G2



OPTUM G3

Future developments

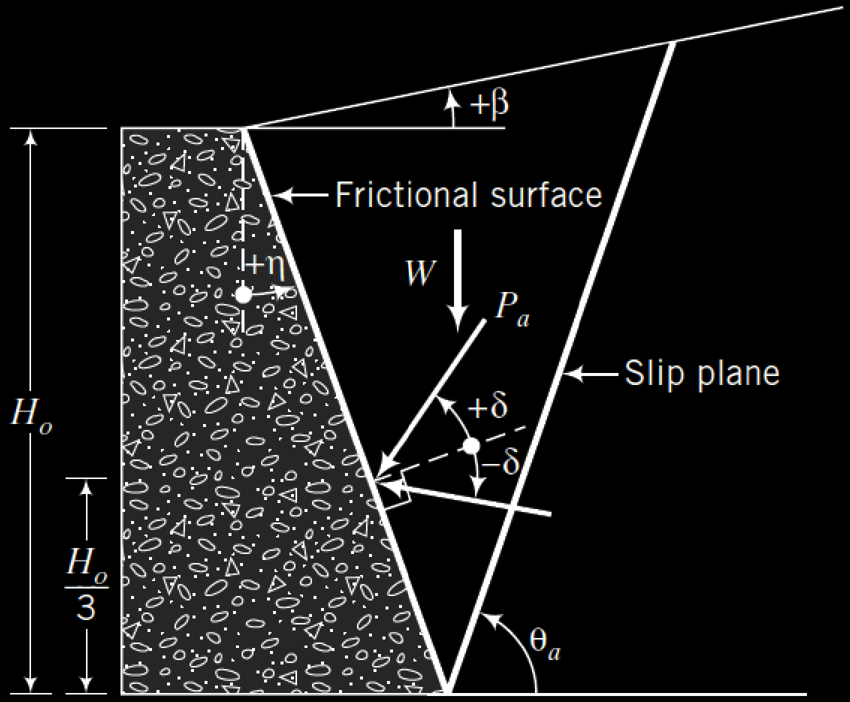
G2 2022

G3 2022

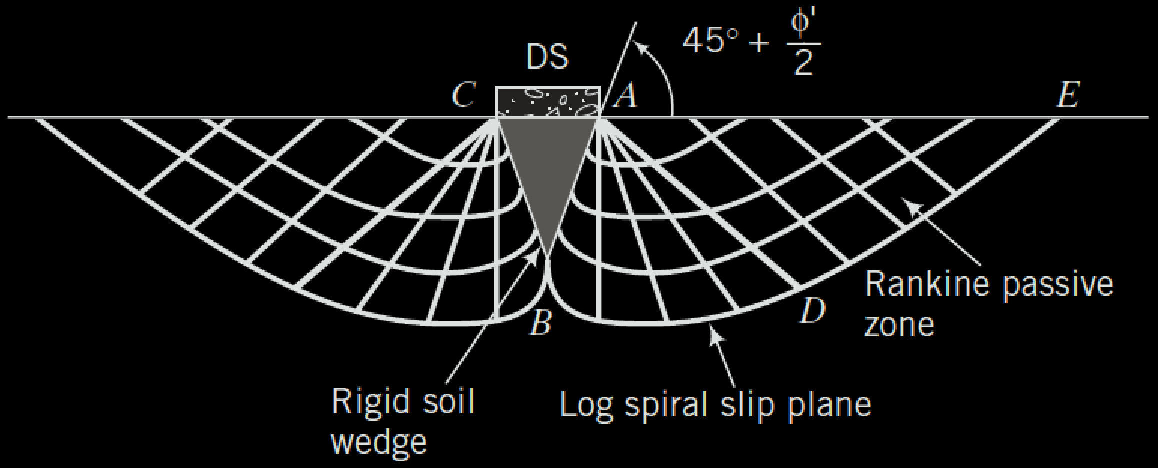
Future developments

The world is 3D – Why do we then use 2D?

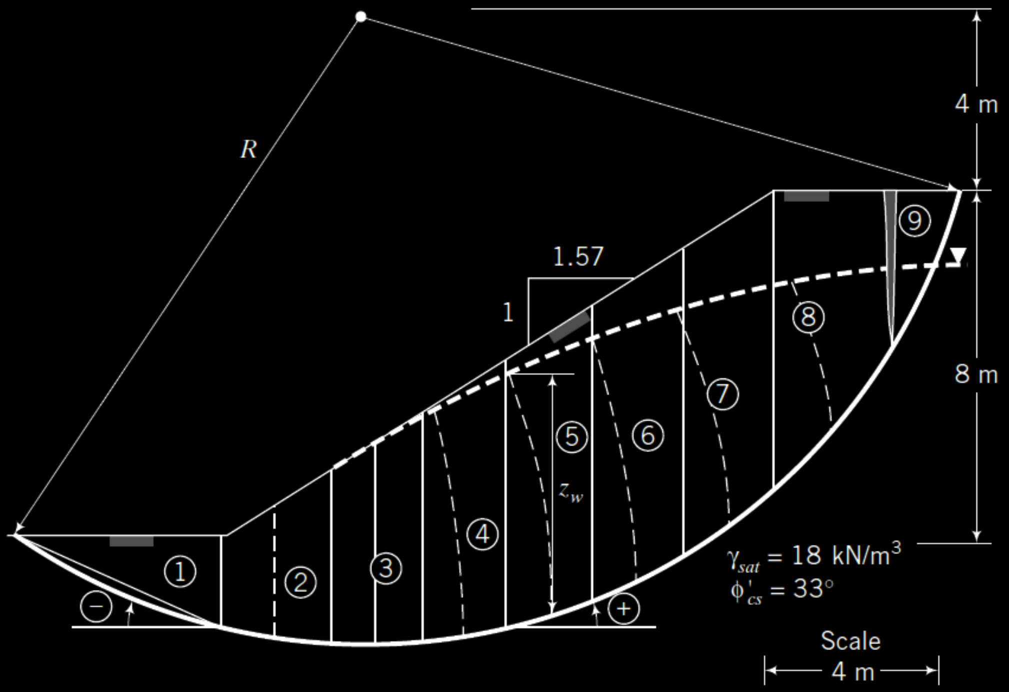
Retaining walls



Foundations

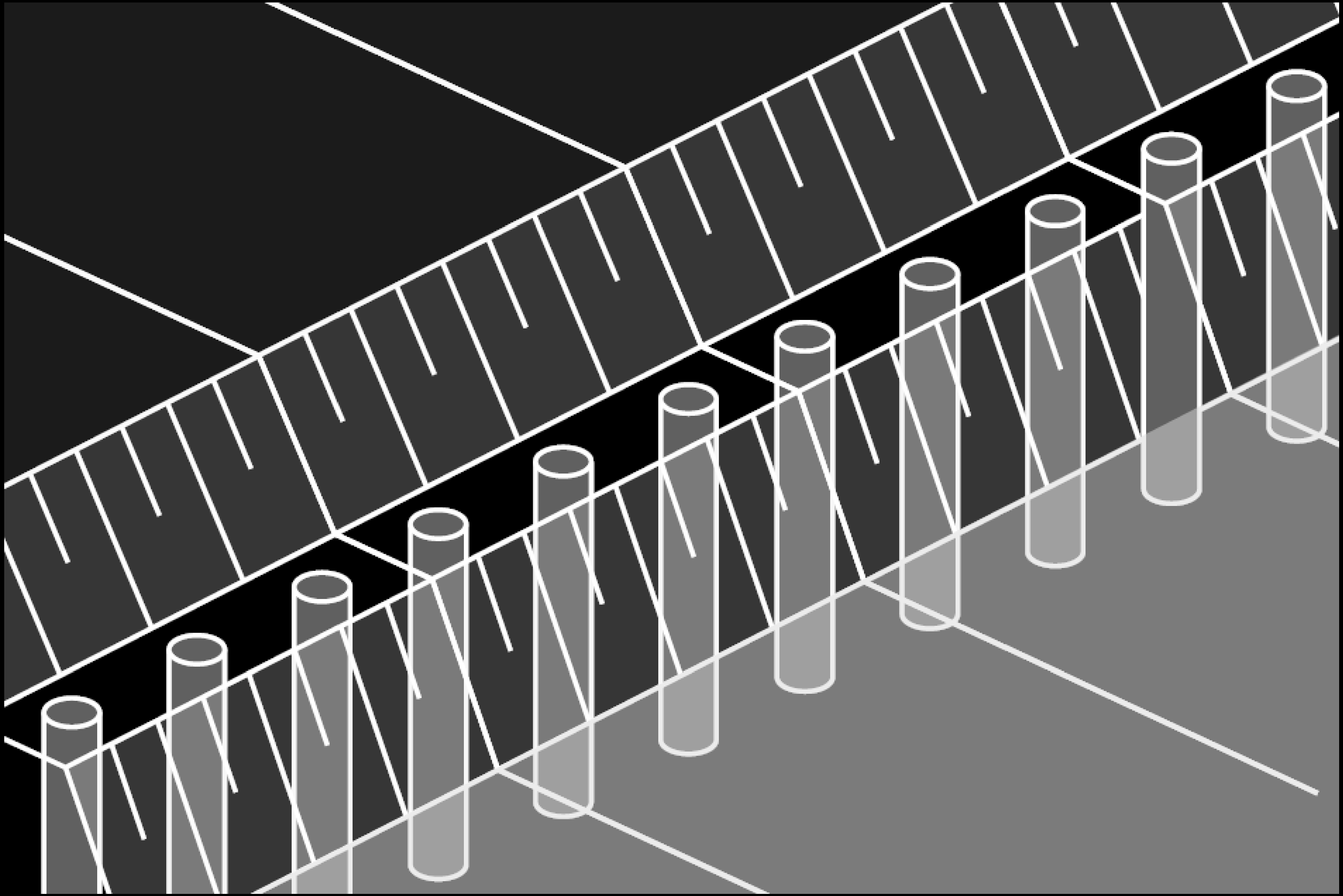


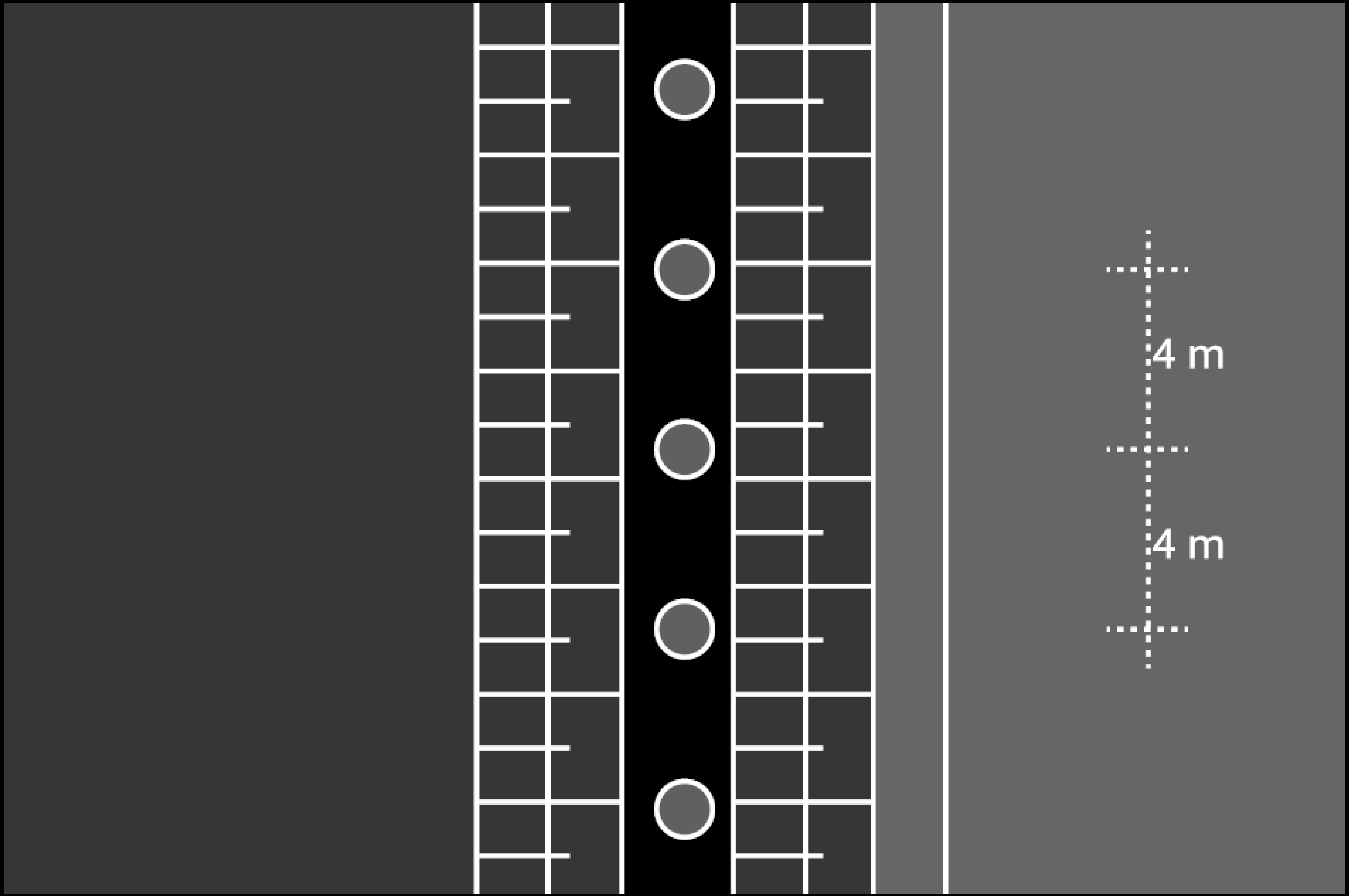
Slopes



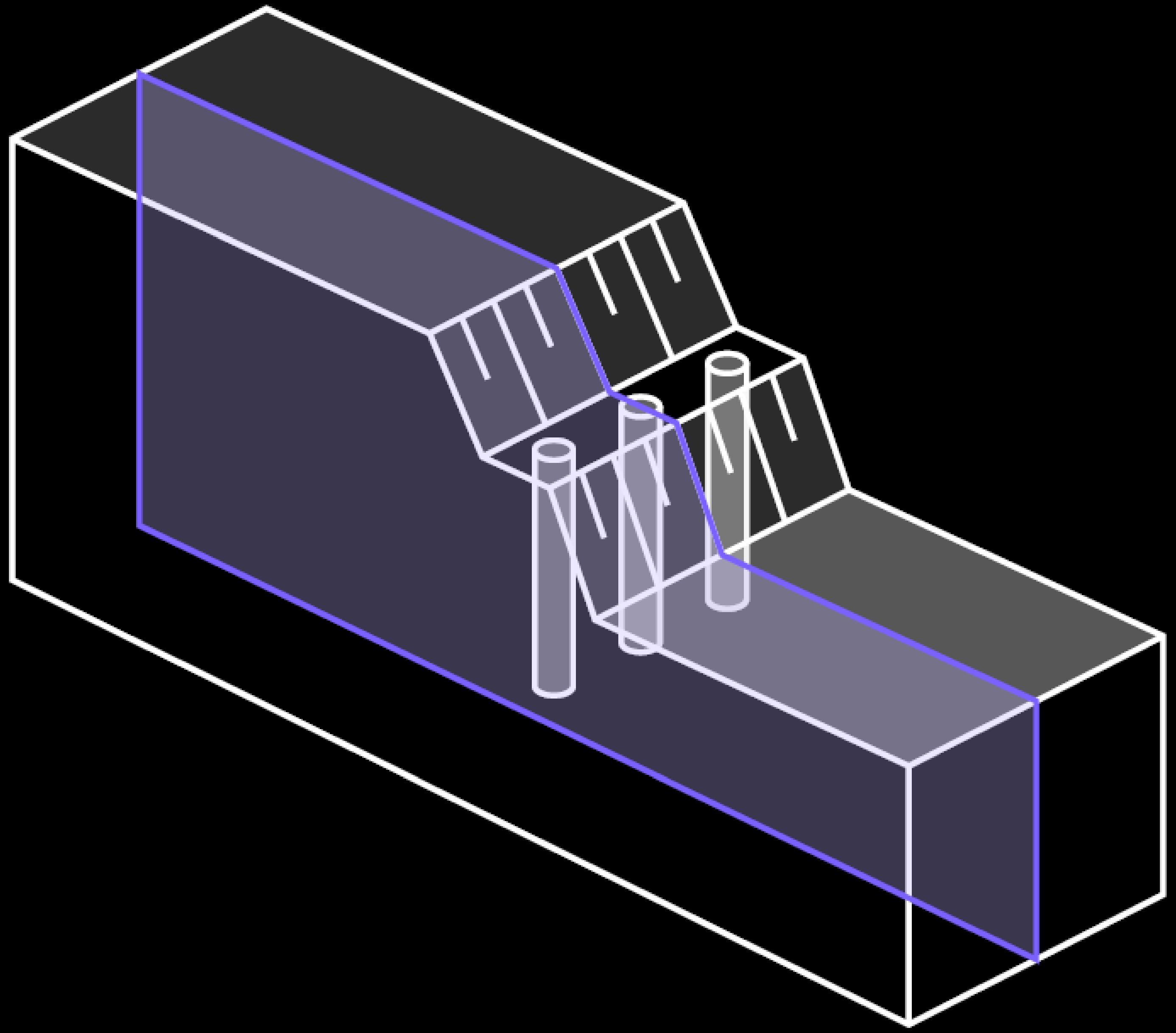
Example 1

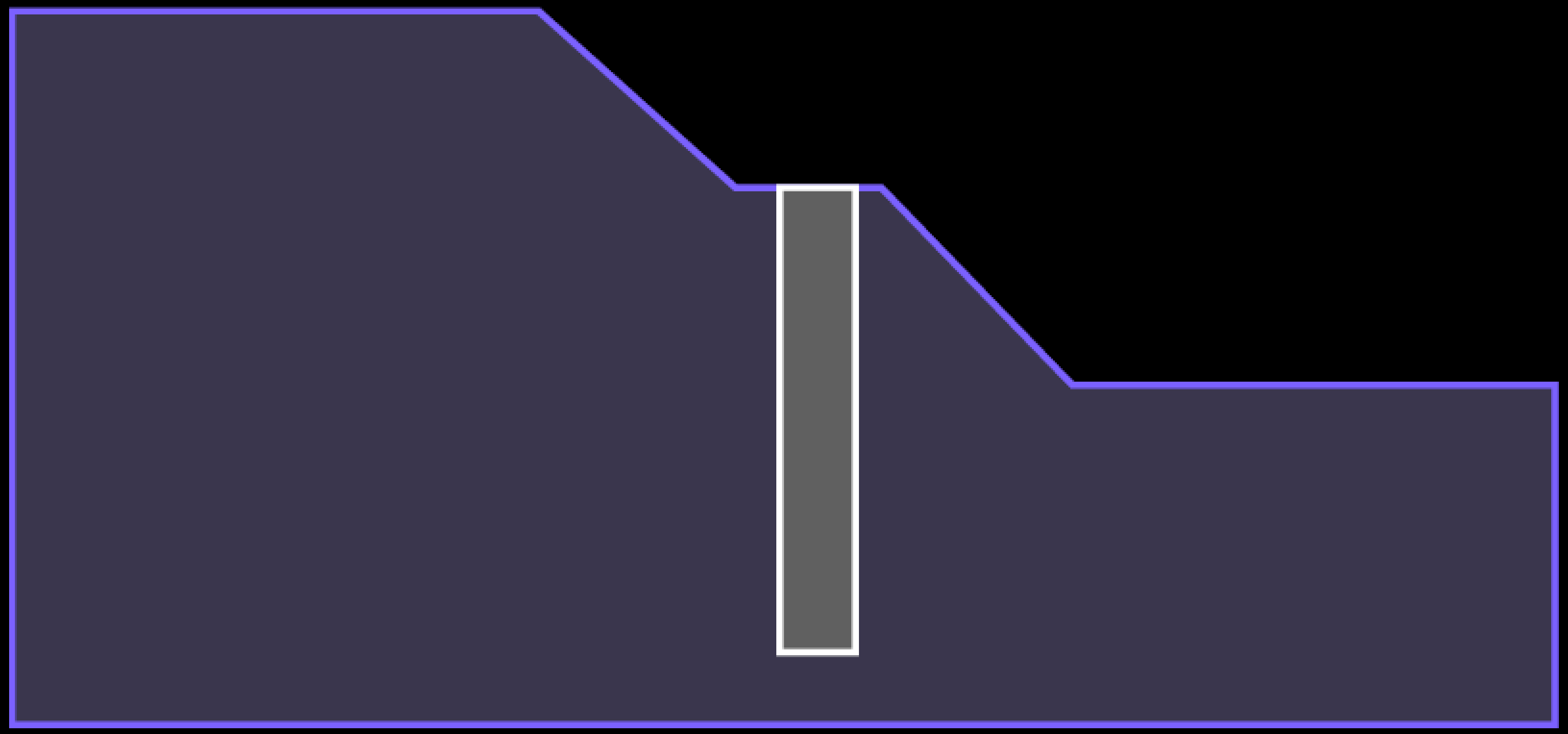
Pile reinforced slope

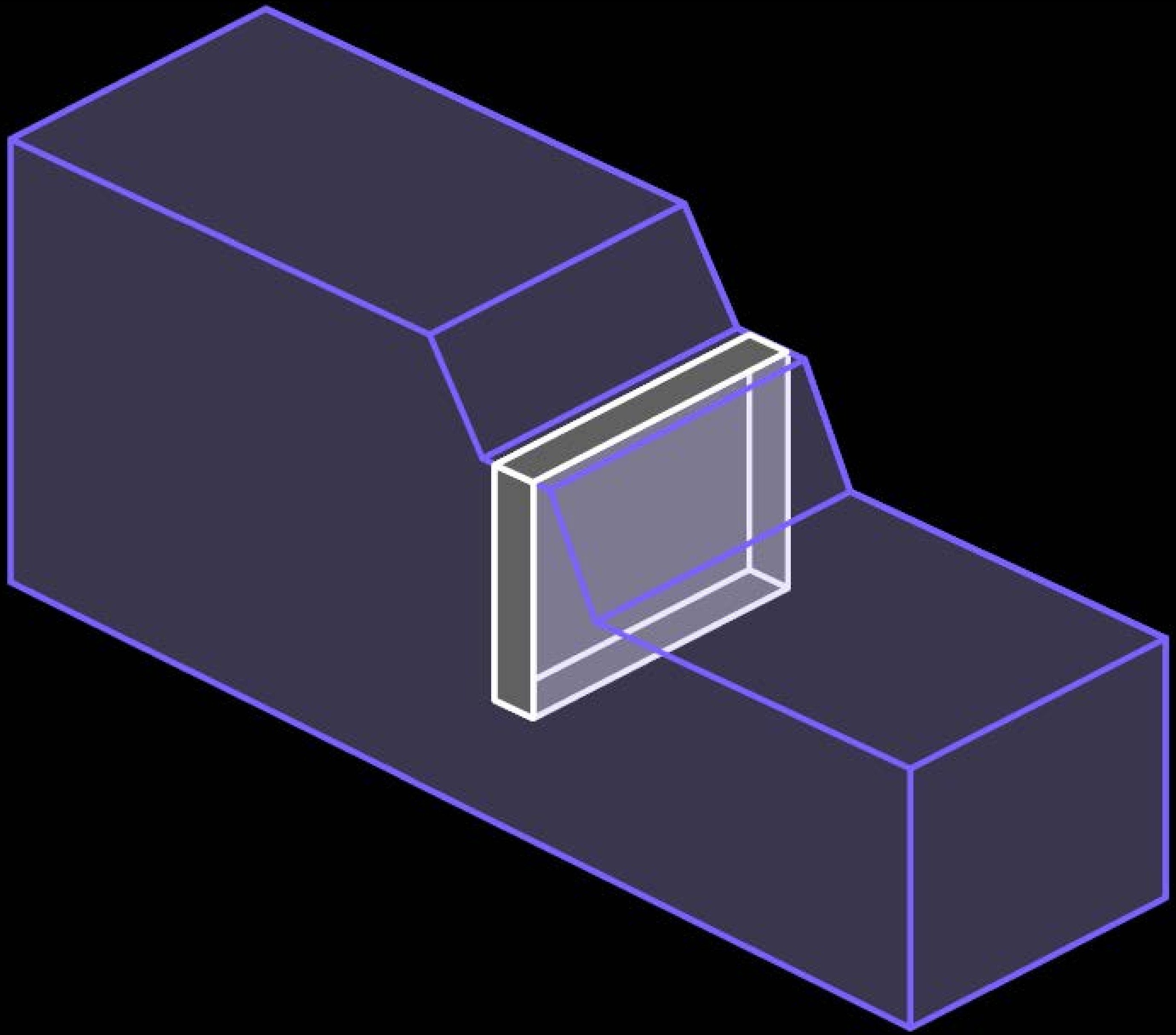




4 m
4 m

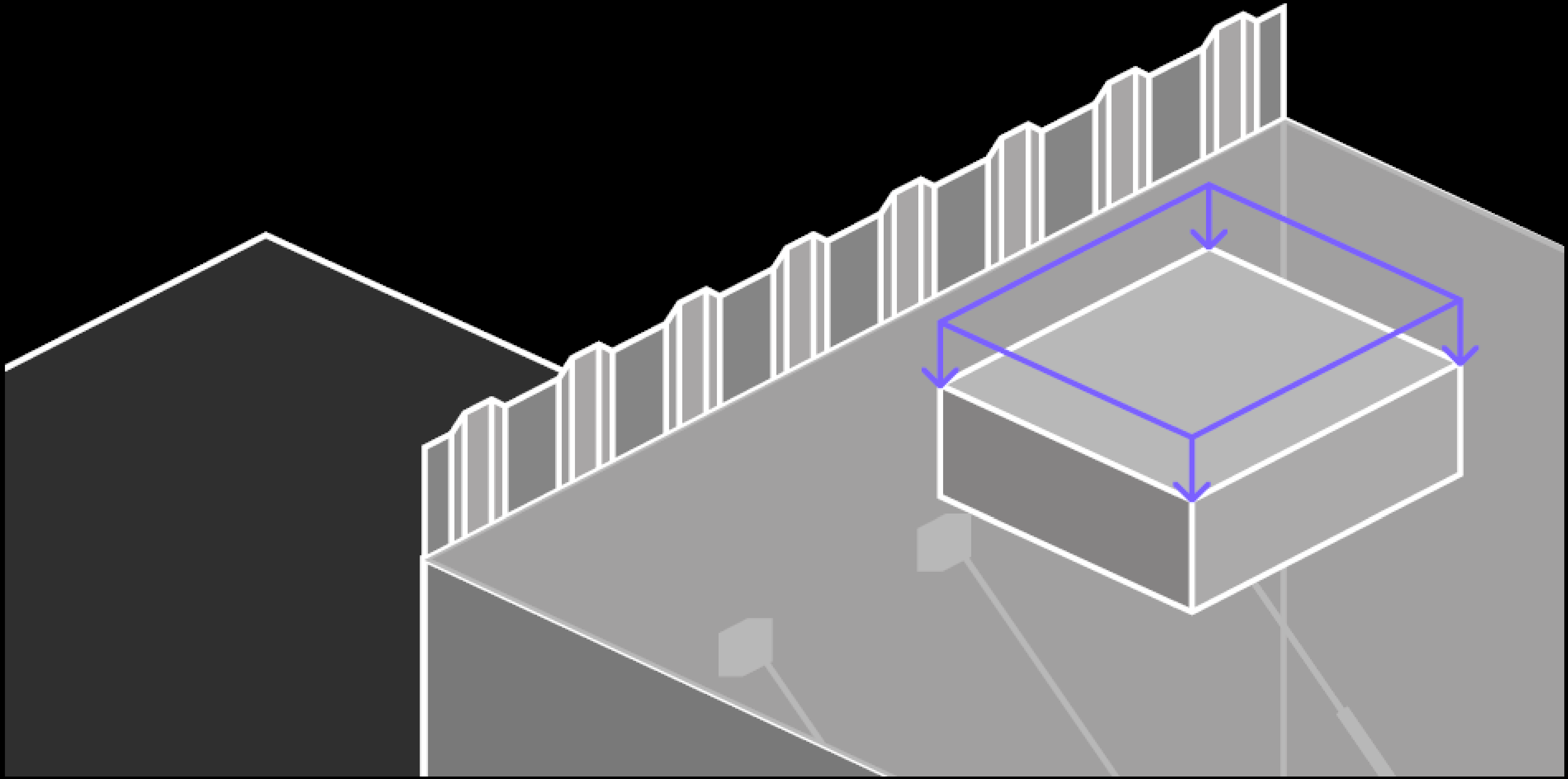


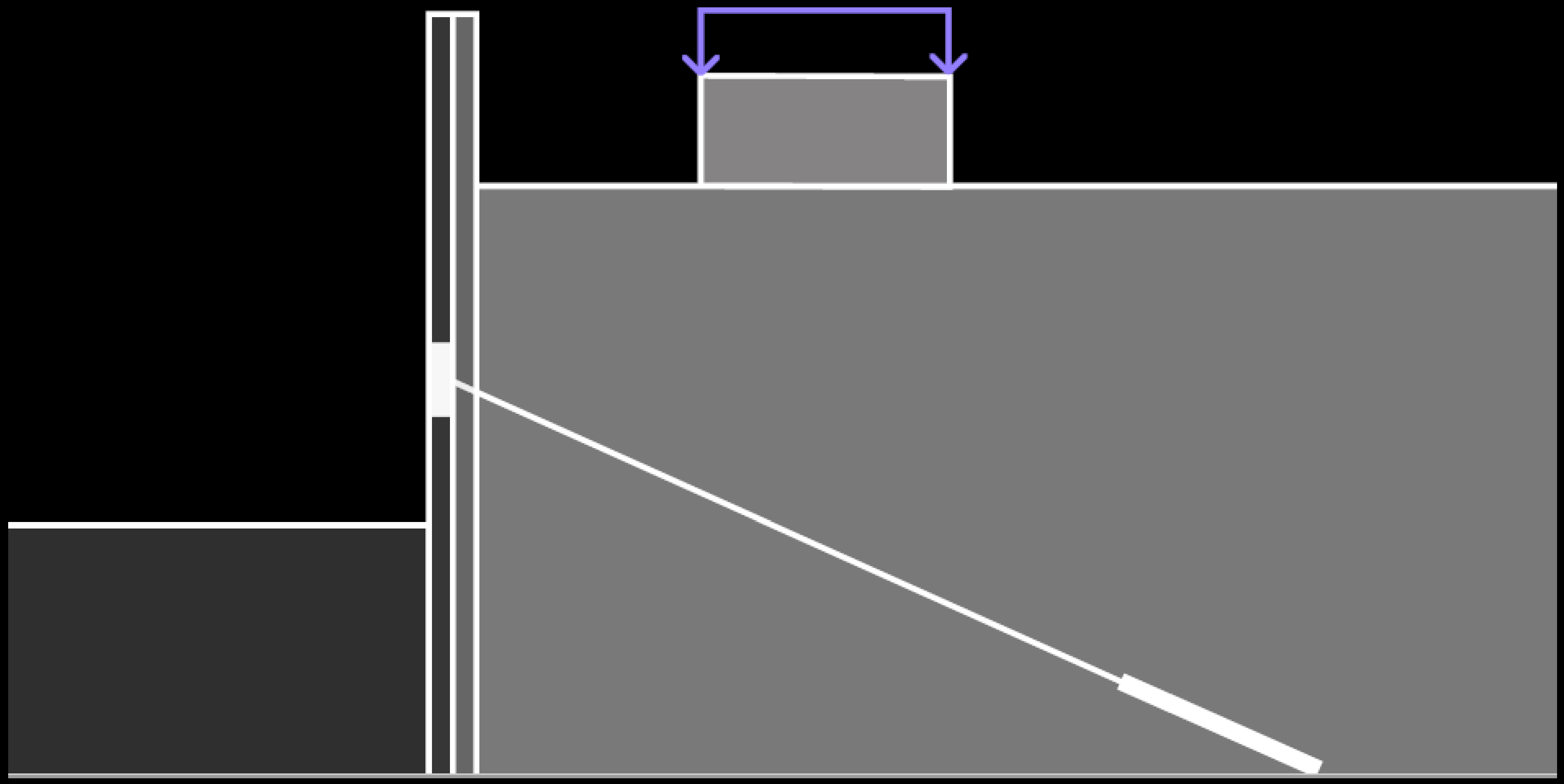




Example 2

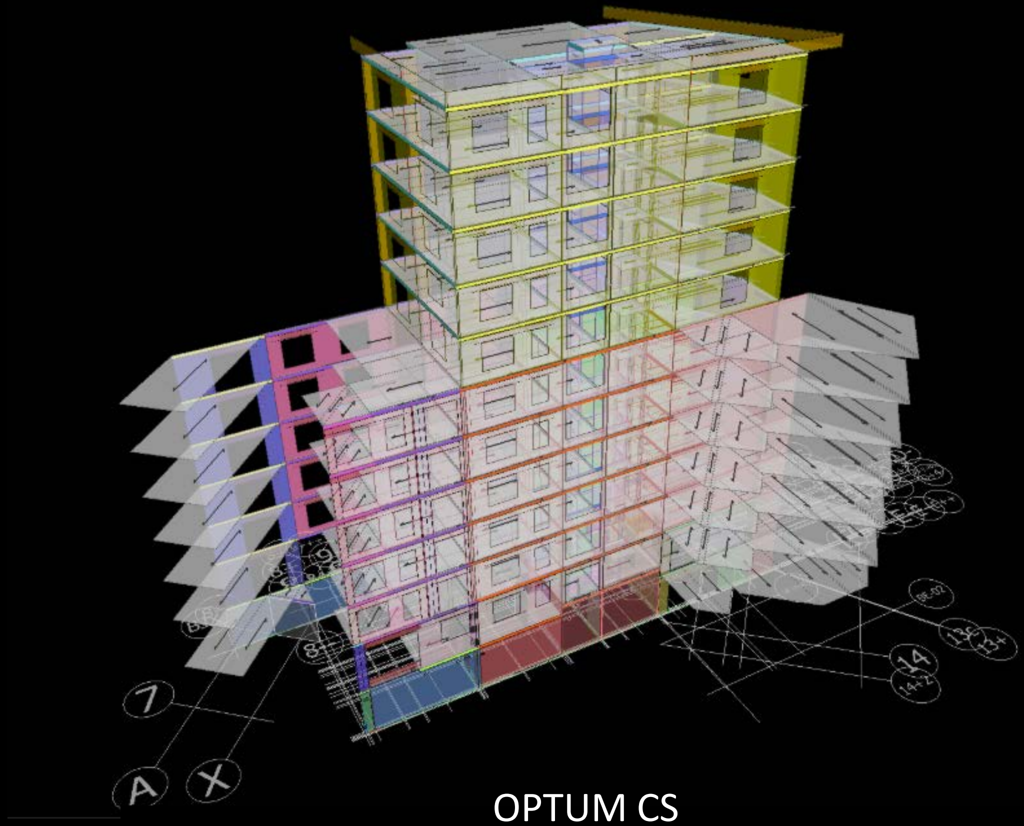
Foundation settlement





Why 2D?

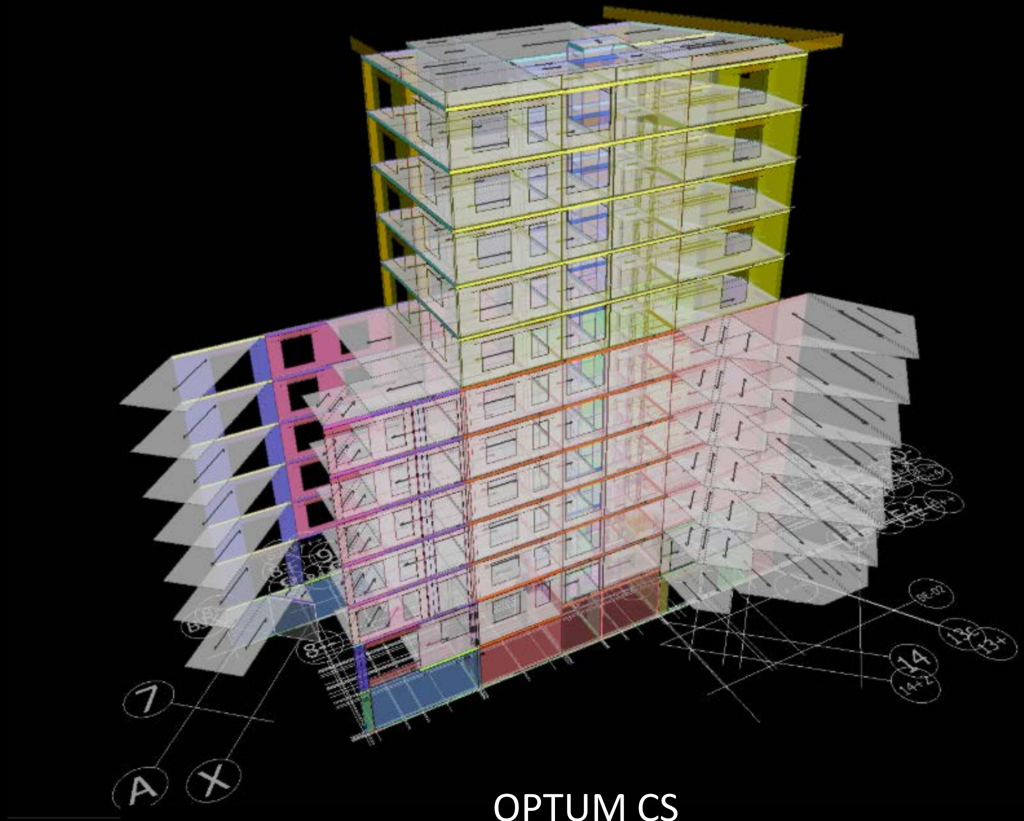
+ All other areas of engineering are in 3D by default – mechanical, aerospace, steel, concrete, etc



OPTUM CS

Why 2D?

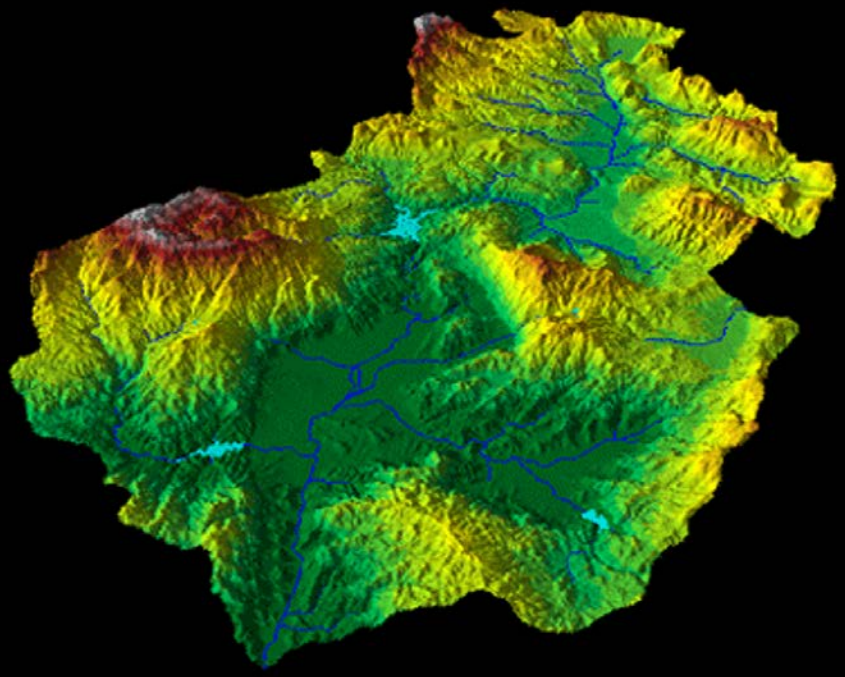
+ All other areas of engineering are in 3D by default – mechanical, aerospace, steel, concrete, etc
+ BIM is inherently 3D – whether or not the calculations are done in 3D



OPTUM CS

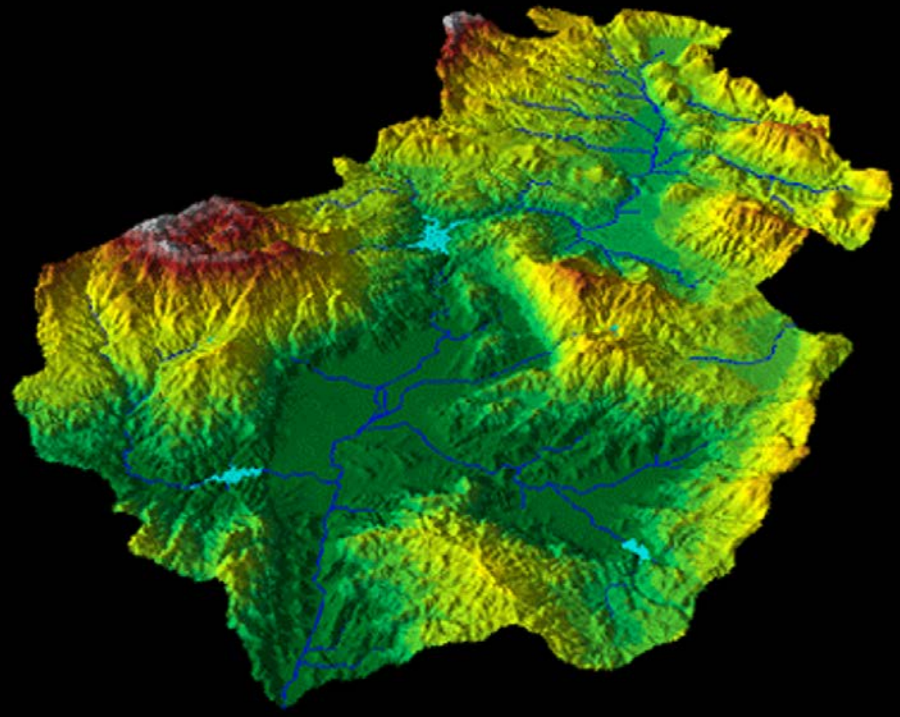
Why 2D?

- All other areas of engineering are in 3D by default – mechanical, aerospace, steel, concrete, etc
- BIM is inherently 3D – whether or not the calculations are done in 3D
- Terrain data increasingly specified in terms of Digitial Elevation Models



On the other hand

+ The world is 3D ***but*** 2D is convenient and sometimes the only feasible approach
+ Many geotechnical problems are to a good approximation 2D
+ Common 2D/3D requirements:
  - Extrude 2D models to 3D
  - Cut through 3D models to obtain 2D sections



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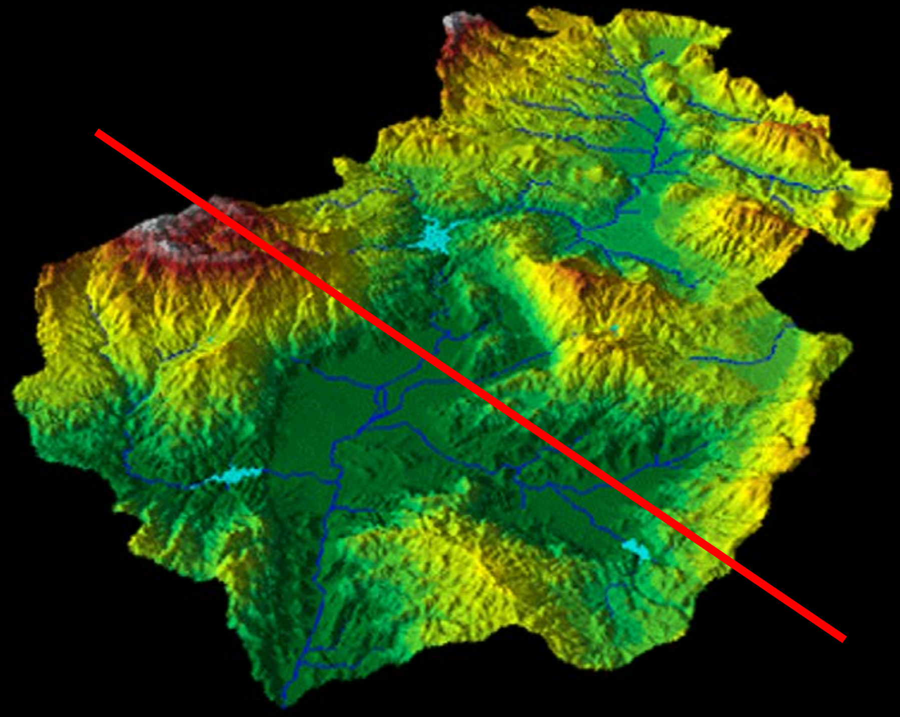

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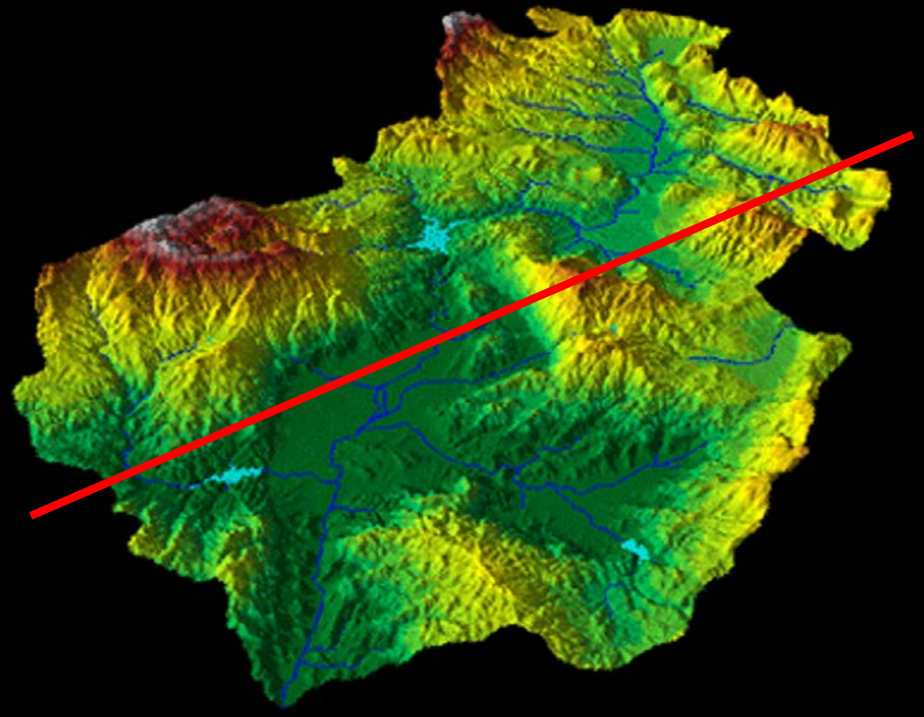

Solution:
Merge 2D and 3D

G2 2022

G3 2022

GX

Optum GX

Optum GX

Thanks for you attention

Strength reduction

Question:

How safe is the slope?

Strength reduction

Question:

What is the ratio between driving and resisting forces?

Strength reduction

Question:

By what factor should the material strength parameters be reduced to induce failure?

Strength reduction

Question:

By what factor should the material strength parameters be reduced to induce failure?

Solve iteratively:

$$\begin{aligned} \text{maximize} \quad & 0 \\ \text{subject to} \quad & \boldsymbol{B}^{\mathsf{T}}\boldsymbol{\sigma} = \boldsymbol{f_0} \\ & F(\boldsymbol{\sigma}, \text{FOS}) \leq 0 \end{aligned}$$

Feasible (stable): increase FOS

Infeasible (unstable): decrease FOS