MODELLING OF LANDSLIDES AS A BIFURCATION PROBLEM

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I Introduction

FAILURE : THE CLASSICAL VIEW

Rate independent materials :

\[ d\sigma = F(d\varepsilon) \]
\[ \forall \lambda \geq 0 : F(\lambda d\varepsilon) = \lambda F(d\varepsilon) \Rightarrow F \text{ Homogeneous of degree 1} \]

Euler’s identity :

\[ d\sigma = F(d\varepsilon) = \frac{\partial F}{\partial (d\varepsilon)} d\varepsilon \quad d\sigma = M_h(v) d\varepsilon \quad \left( v = \frac{d\varepsilon}{|d\varepsilon|} \right) \]

\[ d\sigma_\alpha = M_{\alpha\beta} d\varepsilon_\beta \]

Perfect plasticity :

\[ d\sigma = 0 \quad \text{and} \quad \|d\varepsilon\| \neq 0 \quad \text{(Limit stress state)} \]

\[ \Rightarrow \quad \begin{cases} \det M_h = 0 & \text{Plastic limit condition} \\ M_h d\varepsilon = 0 & \text{Plastic flow rule} \end{cases} \]
Typical behaviour of a loose sand: undrained (isochoric) triaxial compression

- Experimental evidence: $q$ goes through a maximum before Mohr-Coulomb limit
Experimental observations (I. Georgopoulos, J. Desrues – Grenoble, DIGA project):

Partial diffuse failure

Total diffuse failure
A typical example of a diffuse mode of failure:

- $\Delta q = \Delta F/S$: “small” additional force (stress controlled loading)
  - $q$ peak is unstable according to LYAPUNOV definition
  - Non-controllability after R. NOVA definition

- **Second order work criterion**:
  \[
  d^2W = d\sigma_1 d\varepsilon_1 + 2d\sigma_3 d\varepsilon_3 = dq d\varepsilon_1 \quad (1)
  \]

  After HILL condition of stability, $q$ peak is unstable

- **For axisymmetric conditions**:
  \[
  \begin{bmatrix}
  dq \\
  d\varepsilon_v 
  \end{bmatrix} = N
  \begin{bmatrix}
  d\varepsilon_1 \\
  d\sigma_3 
  \end{bmatrix}
  \]

  - Undrained loading: $d\varepsilon_v = 0$
  - Bifurcation criterion: $\det N = 0$
  - At $q$ peak: $dq = 0$
  - Failure rule:
    \[
    N
    \begin{bmatrix}
    d\varepsilon_1 \\
    d\sigma_3 
    \end{bmatrix} = \begin{bmatrix}
    0 \\
    0 
    \end{bmatrix}
    \]

- **Conclusions**: $q$ peak is a proper failure state, strictly inside the plastic limit condition, without localization pattern, properly described by Hill’s condition and a bifurcation criterion
HILL’S CONDITION OF STABILITY

- **DRUCKER’s postulate**
  \[ \forall d\sigma, d\varepsilon^p : d^2W^p = d\sigma : d\varepsilon^p > 0 \]
  always satisfied in associated elasto-plasticity
  \[ d\varepsilon^p = d\lambda (df/d\sigma) \Rightarrow d^2W^p = d\lambda (d\sigma : df/d\sigma) \]
  \[ > 0 \quad > 0 \]

- **HILL’s condition of stability**
  \[ \forall d\sigma, d\varepsilon : d^2W = d\sigma : d\varepsilon > 0 \]
  DRUCKER \(\Rightarrow\) HILL

Incrementally linear constitutive relations
\[ d\sigma = M \, d\varepsilon \quad d^2W = t \, d\varepsilon \, M \, d\varepsilon = t \, d\varepsilon \, M^s \, d\varepsilon \quad \rightarrow \quad d^2W > 0 \iff \det M^s > 0 \]

- **Associated elasto-plasticity**: \( M = M^s \)
  - plasticity limit condition: \( \det M = 0 \)
  - stability condition: \( \det M^s = 0 \) IDENTICAL!

- **Non-associated elasto-plasticity**: \( \det M^s \) is always vanishing before \( \det M \) and \( \det L^1 = 0 \),
  \( \Rightarrow \det M^s = 0 \) is satisfied strictly inside the plastic limit condition and,
  inside the localisation condition
CONSTITUTIVE RELATIONS

\[ d\varepsilon = M(u)d\sigma = M^1d\sigma + \frac{1}{\|d\sigma\|}M^2d\sigma \ d\sigma + \cdots \]

Incrementally non-linear relations of second order:

In fixed stress-strain principal axes:

Incrementally non-linear model (quadratic interpolation):

\[
\begin{align*}
\frac{d\varepsilon_1}{d\sigma_1} &= \frac{1}{2} \left[ N^+ + N^- \right] \frac{d\sigma_1}{d\sigma_1} + \frac{1}{2\|d\sigma\|} \left[ N^+ - N^- \right] \left( \frac{d\sigma_1}{d\sigma_2} \right)^2 + \left( \frac{d\sigma_1}{d\sigma_3} \right)^2 \\
\frac{d\varepsilon_2}{d\sigma_2} &= \frac{1}{2} \left[ N^+ + N^- \right] \frac{d\sigma_2}{d\sigma_2} + \frac{1}{2\|d\sigma\|} \left[ N^+ - N^- \right] \left( \frac{d\sigma_2}{d\sigma_1} \right)^2 + \left( \frac{d\sigma_2}{d\sigma_3} \right)^2 \\
\frac{d\varepsilon_3}{d\sigma_3} &= \frac{1}{2} \left[ N^+ + N^- \right] \frac{d\sigma_3}{d\sigma_3} + \frac{1}{2\|d\sigma\|} \left[ N^+ - N^- \right] \left( \frac{d\sigma_3}{d\sigma_1} \right)^2 + \left( \frac{d\sigma_3}{d\sigma_2} \right)^2
\end{align*}
\]

Octo-linear model (linear interpolation):

\[
\begin{align*}
\frac{d\varepsilon_1}{d\sigma_1} &= \frac{1}{2} \left[ N^+ + N^- \right] \frac{d\sigma_1}{d\sigma_1} + \frac{1}{2\|d\sigma\|} \left[ N^+ - N^- \right] \left( \frac{d\sigma_1}{d\sigma_2} \right)^2 + \left( \frac{d\sigma_1}{d\sigma_3} \right)^2 \\
\frac{d\varepsilon_2}{d\sigma_2} &= \frac{1}{2} \left[ N^+ + N^- \right] \frac{d\sigma_2}{d\sigma_2} + \frac{1}{2\|d\sigma\|} \left[ N^+ - N^- \right] \left( \frac{d\sigma_2}{d\sigma_1} \right)^2 + \left( \frac{d\sigma_2}{d\sigma_3} \right)^2 \\
\frac{d\varepsilon_3}{d\sigma_3} &= \frac{1}{2} \left[ N^+ + N^- \right] \frac{d\sigma_3}{d\sigma_3} + \frac{1}{2\|d\sigma\|} \left[ N^+ - N^- \right] \left( \frac{d\sigma_3}{d\sigma_1} \right)^2 + \left( \frac{d\sigma_3}{d\sigma_2} \right)^2
\end{align*}
\]

Constitutive matrices \( N^+ \) and \( N^- \):

\[
E_i = \left( \frac{\partial \sigma_i}{\partial \varepsilon_i} \right)_{\sigma,\sigma_i}, ~ \nu_i' = -\left( \frac{\partial \varepsilon_i}{\partial \sigma_i} \right)_{\sigma,\sigma_i}
\]

\[
N^+ = \begin{bmatrix} 1 & -v_i^+ & v_i^+ \\ E_1^+ & E_2^+ & E_3^+ \\ -v_i^+ & 1 & v_i^+ \\ E_1^+ & E_2^+ & E_3^+ \\ -v_i^+ & v_i^+ & 1 \\ E_1^+ & E_2^+ & E_3^+ \end{bmatrix}, ~ N^- = \begin{bmatrix} 1 & -v_i^- & v_i^- \\ E_1^- & E_2^- & E_3^- \\ -v_i^- & 1 & v_i^- \\ E_1^- & E_2^- & E_3^- \\ -v_i^- & v_i^- & 1 \\ E_1^- & E_2^- & E_3^- \end{bmatrix}
\]

(+) for compression, (-) for extension

Six generalized triaxial loading paths:

\( (d\sigma_1, d\sigma_2, d\sigma_3) = (\pm 1, 0, 0); (0, \pm 1, 0); (0, 0, \pm 1) \)
BIFURCATION ANALYSIS IN AXISYMMETRIC CONDITIONS

At \((\sigma_1-\sigma_3/R)\) peak : 
\[
\varepsilon_1 + 2R \varepsilon_3 = 0 ; \quad R=\text{cst.}
\]

- \((\sigma_1-\sigma_3/R)\) peak is unstable according to LYAPUNOV
- 
  \[d^2W = d\sigma_1 d\varepsilon_1 + 2 d\sigma_3 d\varepsilon_3 = (d\sigma_1-d\sigma_3/R) d\varepsilon_1 \leq 0\] from \((\sigma_1-\sigma_3/R)\) peak
- 
  \[
  \begin{bmatrix}
  d\sigma_1 - d\sigma_3/R \\
  d\varepsilon_1 + 2R d\varepsilon_3 \\
  d\varepsilon_3 \\
  d\sigma_3/R
  \end{bmatrix}
  = Q
  \begin{bmatrix}
  d\varepsilon_1 \\
  d\sigma_1 \\
  d\sigma_3 \\
  d\sigma_3/R
  \end{bmatrix}
  \]

BIFURCATION CRITERION :

\[
\det Q=0 \iff 2 \frac{E_1}{E_3^2}(1-\nu_3^3)R^2 - 2(\nu_3^1 + E_1^- \nu_3^1^- / E_3^-)R + 1 = 0
\]

\[
\Delta = -\frac{2}{(E_3^-)^2} \det M^s \geq 0
\]

Loss of controllability after NOVA (1994)

First unstable direction : 
\[
\frac{d\sigma_1}{\sqrt{2} d\sigma_3} = \frac{E_1^-}{\sqrt{2 E_3^-}} \frac{(1-\nu_3^3^-)(E_1^- \nu_3^1^- - E_3^- \nu_3^3^-)}{E_1^- (1-\nu_3^3^-) \nu_3^3^- (E_1^- \nu_3^1^- - E_3^- \nu_1^3^-)}
\]

Rupture rule : 
\[
E_1^- d\varepsilon_1 + (2 \nu_3^1^- E_1^- E_3^- - 1/R) d\sigma_3 = 0
\]
BIFURCATION ANALYSIS IN PLANE STRAIN CONDITIONS (H.D.V. KHOA)

- **Second order work:**
  \[
  d^2W = d\sigma_1 d\varepsilon_1 + d\sigma_3 d\varepsilon_3 \\
  = (d\sigma_1 - d\sigma_3/R) d\varepsilon_1 + d\sigma_3 (d\varepsilon_1 + Rd\varepsilon_3)/R \\
  = (d\sigma_1 - d\sigma_3/R) d\varepsilon_1 \leq 0 \quad \text{from } (\sigma_1 - \sigma_3/R) \text{ peak}
  \]

  \[\Rightarrow\] The peak is potentially unstable in Hill’s sense

- **Octo-linear model:**
  \[
  \begin{bmatrix}
  d\sigma_1 - d\sigma_3/R \\
  d\varepsilon_1 + Rd\varepsilon_3
  \end{bmatrix}
  = Q
  \begin{bmatrix}
  d\varepsilon_1 \\
  d\sigma_3/R
  \end{bmatrix}
  \]

  \[\det Q = 0 \quad \text{Bifurcation criterion}\]

  \[\Leftrightarrow E_1^- (1-V_2^3 V_3^2) R^2 - E_3^- (V_3^1 + V_2^2 V_3^1) + E_1^- (V_3^1 + V_2^2 V_3^1) R + E_3^- (1-V_1^2 V_2^1) = 0\]

  \[\Delta = -4 (E_1^-)^2 (E_3^-)^2 \det P^s \geq 0 \quad \begin{bmatrix}
  d\varepsilon_1 \\
  d\varepsilon_3
  \end{bmatrix} = P \begin{bmatrix}
  d\sigma_1 \\
  d\sigma_3
  \end{bmatrix}\]

  \[\det P^s = 0 \quad \text{Bifurcation direction}\]

  \[
  \begin{bmatrix}
  d\sigma_1 \\
  d\sigma_3
  \end{bmatrix}_c = \frac{E_1^- (1-V_2^3 V_3^2)}{E_3^-} \left[ E_1^- (V_3^1 + V_2^2 V_3^1) - E_3^- (V_3^1 + V_2^2 V_3^1) \right]
  \]

  \[
  \begin{bmatrix}
  d\sigma_1 \\
  d\sigma_3
  \end{bmatrix}_c = \frac{E_1^- (1-V_2^3 V_3^2)(1-V_3^3 V_3^2)}{E_3^-} \left[ E_1^- (1-V_1^2 V_2^1)(1-V_3^3 V_3^2) - E_3^- (V_3^3 + V_1^2 V_2^3)^2 \right]
  \]
AXISYMMETRIC PROPORTIONAL STRAIN PATHS – INL2 model, DENSE SAND

$$R \in (0.3, 0.35, 0.4, 0.45, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0)$$

Volumetric strain versus axial strain on the left and deviatoric stress versus mean pressure on the right

Deviatoric stress versus axial strain on the left and $$(\sigma_1 - \sigma_3/R)$$ versus axial strain on the right
Loose Hostun sand $\Rightarrow$ liquefaction for all R values

Dense Hostun sand $\Rightarrow$ liquefaction for only low R values

Proportional strain paths simulated by the incrementally octo-linear model for different R values ($R=[0.1,0.2,0.3,0.35,0.4,0.45,0.5,0.6,0.7,0.8,0.9,1.0]$)
Instability domain for the dense sand in axisymmetric conditions.

\sigma_1- \sqrt{2}\sigma_3 plane in the cases of the octo-linear model (+) and the non linear one (●).

First stress directions giving a nil \(d^2W\) for the dense sand in the \(\sigma_1- \sqrt{2}\sigma_3\) plane, with the octo-linear model (+) and the non linear one (●).
BIFURCATIONS IN AXISYMMETRIC CONDITIONS

Cones of unstable stress directions: Dense sand; non-linear model
Bifurcation domains and first bifurcation directions corresponding to:

- **vanishing** $d^2 W$
- **vanishing** $d^2 W$ (red) or $\text{det} P_s$ (blue)

Bifurcation domains are **larger** for the case of loose Hostun sand than for the dense one.

The above results of two bifurcation indicators are **completely numerically coincide**.
Boundary of the bifurcation domain and the first incremental stress directions of vanishing $d^2W$, according to Hill’s condition, for dense Hostun sand (left) and loose Hostun sand (right). Diagrams in the $(\sigma_1, \sigma_3)$ plane ($\varepsilon_2 = 0$) in the case of the octo-linear constitutive model.
BIFURCATION DOMAIN (3D)

Stress paths in the deviatoric plane

Instability surface for the dense sand (tr $\sigma = 300$ kPa)

Instability surface for the loose sand (tr $\sigma = 300$ kPa)

Stress states of stability analysis
THE MICRO-DIRECTIONAL MODEL (F. NICOT)


\[
\begin{align*}
\dot{u}_i (\theta, \varphi) & \approx 2r_g \; d\varepsilon \; n_j (\theta, \varphi) \\
\dot{F}_n & = k_n \; d\dot{u}_n \\
\dot{F}_i & = \min \left\{ \left\| \dot{F}_i + k_t \; d\dot{u}_i \right\|, \tan \varphi_g \left( \dot{F}_n + k_n \; d\dot{u}_n \right) \right\} \\
\sigma_{ij} & = \frac{2r_g}{\nu_e} \sum_c \dot{F}_c \; n_j
\end{align*}
\]

(Contact direction)

\[n(\theta, \varphi)\]

\[\epsilon_d \Rightarrow \sigma, \varepsilon\]

Strain localisation operator

Local behaviour

Stress averaging

(Chang, 1992; Cambou, 1993, etc)
Probing tests and normalised second order work

General expression of the second order work

\[ d^2W = d\bar{\sigma} \, d\bar{\varepsilon} \]

Expression of the second order work in axisymmetric conditions

\[ d^2W = d\sigma_1 \, d\varepsilon_1 + 2 \, d\sigma_2 \, d\varepsilon_2 \]

Normalised second order work

\[ d^2w = \frac{d^2W}{\sqrt{d\sigma_1^2 + 2 \, d\sigma_2^2 \, \sqrt{d\varepsilon_1^2 + 2 \, d\varepsilon_2^2}}} \]
Polar diagrams of the normalised second order work

Axisymmetric case

Micro-mechanical interpretation

\[ d^2 W = d^2 W_l + d^2 W_v + d^2 W_f \]

- **Local second order work**
  \[
  d^2 W_l = \frac{3}{2\pi N_g r_g^2} \frac{\rho_o}{\rho_g} \int_0^{\pi} \int_{\theta} \hat{\mathcal{F}}(\theta, \varphi) d\hat{u} \omega(\theta, \varphi) \sin \varphi d\theta d\varphi
  \]

- **Change in volume**
  \[
  d^2 W_v = \frac{3}{2\pi N_g r_g^2} \frac{\rho_o}{\rho_g} \int_0^{\pi} \int_{\theta} \hat{\mathcal{F}}(\theta, \varphi) d\hat{u} \omega(\theta, \varphi) \sin \varphi d\theta d\varphi
  \]

- **Change in fabric**
  \[
  d^2 W_f = \frac{3}{2\pi N_g r_g^2} \frac{\rho_o}{\rho_g} \int_0^{\pi} \int_{\theta} \hat{\mathcal{F}}(\theta, \varphi) d\hat{u} \omega(\theta, \varphi) \sin \varphi d\theta d\varphi
  \]
Micro-mechanical interpretation

In the plastic case

\[ d^2 \hat{W} = d \hat{F} \cdot \hat{\mu} = d \hat{F}_n \, d \hat{u}_n + d \hat{F}_{t_1} \, d \hat{u}_{t_1} + d \hat{F}_{t_2} \, d \hat{u}_{t_2} \]

Quadratic form of the second order work

\[ d^2 \hat{W} = k_n \, d \hat{u}_n^2 + \tan \phi_g \, \cos \alpha \, k_n \, d \hat{u}_n \, d \hat{u}_i + k_t \, \sin^2 \alpha \, d \hat{u}_i^2 \]

\[ d \hat{u}_n \leq 0 \]

\[ \tan \alpha \leq \frac{\tan \phi_g \, \sqrt{k_n}}{2} \]

\[ \alpha \in \left[ -\frac{\pi}{2} ; \frac{\pi}{2} \right] \]

\[ d \hat{u}_i \in \left[ U_1 ; U_2 \right] \]

\[ U_i = \frac{1 + \xi_i \sqrt{1 - 4 \frac{k_t \tan^2 \alpha}{k_n \tan^2 \phi_g}}}{2 k_t \sin^2 \alpha} \]

\[ \xi_1 = -1 \quad \xi_2 = 1 \]
Discrete analysis of stability from D.E.M. computations (L. Sibille)

The Discrete Element Model:

- SDEC software (Donzé & Magnier 1997): molecular dynamics approach such as Cundall's one (1979).

- Contact interaction defined by 3 mechanical parameters:

  \[ 10^6 \leq k_n \leq 10^6 \, N/m \]
  \[ k_t = k_n / 0.42 \]
  \[ \Phi_{cont} = 35^\circ \]

- Cubic form of the specimen; 10,000 spheres; continuous size distribution; dense specimen:

  \[ 2 \leq D_{sphere} \leq 9 \, mm \]

  For \( p = 100 \, kPa \): \( n = 0.38 \)

- All paths in principal stress (\( \sigma_1, \sigma_2, \sigma_3 \)) or strain (\( \varepsilon_1, \varepsilon_2, \varepsilon_3 \)) spaces are allowed. In this lecture: axi-symmetric conditions only.
Stress probes:

- The initial axi-symmetric stress states:
  1. isotropic compression (100, 200 or 300 kPa),
  2. triaxial drained compression \((\sigma_2 = \sigma_3 = \text{cst.})\)
    characterised by:
    \[
    q = \frac{\sigma_1 - \sigma_3}{(\sigma_1 + \sigma_2 + \sigma_3)/3}
    \]

- The loading program:
  - defined in the Rendulic plane of stress increments,
  - \(|d\sigma| = \sqrt{(d\sigma_1)^2 + (\sqrt{2}d\sigma_3)^2} = \text{cte} = 1\text{kPa}
  - \(0 \leq \alpha_{d\sigma} < 360^\circ\)
    \(\rightarrow\) response vectors \(d\varepsilon\) defined in dual plane \((\sqrt{2}d\varepsilon_3, d\varepsilon_1)\).
  - \(d^2W = d^2W(\alpha_{d\sigma})\)
Discrete analysis of stability from D.E.M. computations

Unstable directions:

Diagram of \( f(\alpha_{d\sigma}) = d^2W_{\text{norm.}} + c \) with:

\[
\begin{aligned}
    d^2W_{\text{norm.}} &= \frac{d\sigma \cdot d\varepsilon}{\|d\sigma\|d\varepsilon} \\
    c \text{ such as: } f(\alpha_{d\sigma}) &> 0
\end{aligned}
\]

\( \sigma_3 = 100 \text{ kPa} \)

\( \sigma_3 = 200 \text{ kPa} \)

→ Cones of "unstable" stress directions observed with the D.E.M.
Discrete analysis of stability from D.E.M. computations

Cones of unstable stress directions: D.E.M. and Incremental Non-Linear model

Discrete Element Model

- $d^2W > 0$
- Cones of unstable directions (L. Sibille)

Macroscopic phenomenological relation (I.N.L.2 model)
2nd step – from the bifurcation point to the failure

- *Imposimato and Nova (1998)* shown that the full controllability of a loading programme defined by its control parameters can be lost before reaching the plastic limit condition.
- The control parameters can be linear combinations of stresses or strains (e.g. volumetric strains).

Loose specimen, $\eta = 0.46$, $\alpha = 215.3$ deg $(q = \text{cst})$

- Stress probe is fully stress controlled $\rightarrow$ no failure observed.
- Can we choose others control parameters?
  
  \[ d^2W = d\sigma_1 d\varepsilon_1 + 2 d\sigma_3 d\varepsilon_3 = dq d\varepsilon_1 + d\sigma_3 d\varepsilon_v \]
- Can we control the loading programme defined by:
  
  \[ dq = 0 \text{ and } d\varepsilon_v = -0.002 \text{ % (dilatancy)} \]

\[ f(\alpha) = d^2W_{\text{norm.}} + c \]
Comparison of the response of the numerical specimen controlled by:

\[ dq = 0 \text{ and } d\sigma_3 < 0 \]

or by \( dq = 0 \text{ and } d\varepsilon_v < 0 \)

\[ \Rightarrow \text{Loss of controllability for } dq = 0 \text{ and } d\varepsilon_v < 0, \]
the specimen totally collapses.
2nd step – from the bifurcation point to the failure

Generalization to proportional stress paths

\[ d^2W = d\sigma_1 d\varepsilon_1 + 2d\sigma_3 d\varepsilon_3 \]

\[ d^2W = d\varepsilon_1 \left( d\sigma_1 - \frac{d\sigma_3}{R} \right) + \frac{d\sigma_3}{R} \left( d\varepsilon_1 + 2R d\varepsilon_3 \right) \]

with \( R = \text{cst.} \) for a given loading path.

Can we control the loading programme defined by:

\( d\sigma_1 - d\sigma_3 / R = 0 \) and \( d\varepsilon_1 + 2R d\varepsilon_3 = -0.002\% \)?

with: \( R = 1.94 \) \( (\alpha = 200^\circ, \ d^2W > 0) \)

\( R = 1.22 \) \( (\alpha = 210^\circ, \ d^2W < 0) \)

\( R = 1.00 \) \( (\alpha = 215.3^\circ, \ d^2W < 0) \)

\( R = 0.843 \) \( (\alpha = 220^\circ, \ d^2W < 0) \)

\( R = 0.593 \) \( (\alpha = 230^\circ, \ d^2W < 0) \)

\( R = 0.408 \) \( (\alpha = 240^\circ, \ d^2W > 0) \)

\[ f(\alpha) = d^2W_{\text{norm.}} + c \]
2nd step – from the bifurcation point to the failure

- The loading programme is controllable for $R = 1.94$ and $0.408$ ($d^2W > 0$).
- Loss of controllability for $R = 1.00$; $0.843$; $0.593$ ($d^2W < 0$).
- For $R = 1.22$ ($d^2W < 0$ but stress direction close to the border of the cone) the loss of controllability is not total.
2\textsuperscript{nd} step – from the bifurcation point to the failure

- For $R = 1.94$ and 0.408 ($d^2W > 0$) a new stable state close to the initial one is reached.
- Total collapse for $R = 1.00$; 0.843; 0.593 ($d^2W < 0$).
- For $R = 1.22$ ($d^2W < 0$ but stress direction close to the border of the cone ) the collapse is partial.

⇒ Loading parameters exist such as the specimen collapses from a bifurcation point detected by the sign of $d^2W$. 
2nd step – from the bifurcation point to the failure

In their tests *Chu et al. 2003* do not apply any loading path. They verify if a mechanical state governed by specific control parameters can be sustained (*Notion of loss of sustainability, Nicot and Darve 2006*).

- **Control of the mechanical state by:**
  
  \[ d\sigma_1 - d\sigma_3 / R = 0 \quad \text{and} \quad d\varepsilon_1 + 2R d\varepsilon_3 = 0 \]

  with:
  
  \[ R = 4.01 \quad (\alpha = 190°, \ d^2W > 0) \]
  \[ R = 1.94 \quad (\alpha = 200°, \ d^2W > 0) \]
  \[ R = 1.22 \quad (\alpha = 210°, \ d^2W < 0) \]
  \[ R = 1.00 \quad (\alpha = 215.3°, \ d^2W < 0) \]
  \[ R = 0.843 \quad (\alpha = 220°, \ d^2W < 0) \]
  \[ R = 0.593 \quad (\alpha = 230°, \ d^2W < 0) \]
  \[ R = 0.408 \quad (\alpha = 240°, \ d^2W > 0) \]
  \[ R = 0.257 \quad (\alpha = 250°, \ d^2W > 0) \]

- **Small perturbation of the numerical specimen**

  External input of kinetic energy (1 \(10^{-5}\) J) to the specimen by excitements of some floating grains (simulation without gravity).
2\textsuperscript{nd} step – from the bifurcation point to the failure

- For $R = 4.01; 1.94; 0.408; 0.257$ ($d^2W > 0$) a new stable state is reached.
- Total collapse for $R = 1.00; 0.843; 0.593$ ($d^2W < 0$).
- For $R = 1.22$ ($d^2W < 0$ but stress direction close to the border of the cone) the collapse is partial.
CONCLUSIONS

1. **Phenomenological analysis**: for non associated materials like geomaterials, there is not a single plastic limit surface where failure occurs, but rather a whole domain in the stress space where bifurcations, losses of uniqueness, instabilities … i.e. FAILURES can appear, according to:
   - the stress-strain history
   - the current direction of loading
   - the loading mode

   In this bifurcation domain, various failure modes can develop (material instabilities leading to diffuse or localized failures, geometric instabilities, …)

   Second order work criterion seems to detect diffuse failure

2. **Micromechanical analysis**: it confirms these analyses. Moreover a new micro-mechanical understanding of these material instabilities is proposed by considering the local, discrete, second order work at the grain level.

3. **Discrete element analysis**: bifurcation domains and cones of unstable stress-strain directions also exhibited in good qualitative agreement. Diffuse failure was simulated exactly for the conditions predicted by the theory.

These 3, basically different, methods give similar results
Some recent contributions

**Book**: Degradations and Instabilities in Geomaterials, F. Darve and I. Vardoulakis eds, Springer publ., 367 pages, 2005

**Papers**:
II Landslides modelling: Trévoux and Petacciato examples

FINITE ELEMENT MODELLING OF THE TREVOUX AND PETACCIATO LANDSLIDES BY TAKING INTO ACCOUNT UNSATURATED HYDRO-MECHANICAL COUPLING
Application to Trévoux and Petacciato landslides

**Trévoux**

- Flood level
- Usual level
- Conventional landslide surface
- Real landslide surfaces
- Intermediate slope landslide

Trévoux hillside (France, 1983)

**Petacciato**

- Deep crack (square in front of the parish church).
- "Vaccareggia", crack on the road near the slide boundary.
The Trevoux site

Geographic location of the studied area
The Trevoux site

Typical cross section of the Trevoux landslide

Finite element mesh of the Trevoux landslide
Geographic location of the studied area

The Petacciato site
The Petacciato site

Typical cross-section of the Petacciato landslide

Finite element mesh of the Petacciato coastal slope
Description of the Plasol constitutive model  
(Liège university)

- Van Eekelen yield criterion:

\[ f = I_{2\sigma} + m \left( I_\sigma - \frac{3c}{\tan \varphi_c} \right) = 0 \]

- With:

\[ m = a \left( 1 + b \sin 3\varphi \right)^n \]

\[ b = \left( \frac{r_c}{r_e} \right)^{\frac{1}{n}} - 1 \]

\[ a = \frac{r_c}{(1+b)^n} \]

\[ r_c = \frac{2 \sin \varphi_c}{\sqrt{3(3 - \sin \varphi_c)}} \]

\[ r_e = \frac{2 \sin \varphi_e}{\sqrt{3(3 + \sin \varphi_e)}} \]

\[ n = -0.229 \]
Description of the Plasol constitutive model

- **evolution of the internal variables:**
  - hyperbolic function of:
    \[
    \varepsilon_{eq}^p = \int_0^t \dot{\varepsilon}_{eq}^p \, dt = \int_0^t \frac{2}{3} \text{tr}(\dot{\varepsilon})^2 \, dt
    \]
    \[
    \dot{\varepsilon} = \dot{\varepsilon}_{eq}^p - \frac{\text{tr}(\dot{\varepsilon}_{eq}^p)}{3} \varepsilon_{eq}^p
    \]
  - Internal variables:
    \[
    \varphi_c = \varphi_{c0} + \left(\varphi_{cf} - \varphi_{c0}\right) \frac{\varepsilon_{eq}^p}{B_p + \varepsilon_{eq}^p}
    \]
    \[
    \varphi_e = \varphi_{e0} + \left(\varphi_{ef} - \varphi_{e0}\right) \frac{\varepsilon_{eq}^p}{B_p + \varepsilon_{eq}^p}
    \]
    \[
    c = c_0 + \left(c_f - c_0\right) \frac{\varepsilon_{eq}^p}{B_c + \varepsilon_{eq}^p}
    \]

\[(\varphi_0 = 30^\circ, \varphi_f = 35^\circ)\]
### Trévoux soil parameters

<table>
<thead>
<tr>
<th>Soil parameters</th>
<th>Unit</th>
<th>Upper fill (1)</th>
<th>Low fill (2)</th>
<th>Sand clay (3)</th>
<th>Gravelly sand (4)</th>
<th>Gravelly marl (5)</th>
<th>Compact marl (6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grain specific weight</td>
<td>kN/m³</td>
<td>26.</td>
<td>26.</td>
<td>29.00</td>
<td>28.00</td>
<td>27.</td>
<td>28.5</td>
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<tr>
<td>Young modulus</td>
<td>MPa</td>
<td>38.6</td>
<td>38.6</td>
<td>30.0</td>
<td>20.0</td>
<td>46.3</td>
<td>100.0</td>
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<tr>
<td>Poisson’s ratio</td>
<td></td>
<td>0.29</td>
<td>0.29</td>
<td>0.29</td>
<td>0.29</td>
<td>0.29</td>
<td>0.29</td>
</tr>
<tr>
<td>Porosity</td>
<td></td>
<td>0.3</td>
<td>0.3</td>
<td>0.5</td>
<td>0.5</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Intrinsic permeability</td>
<td>m²</td>
<td>10⁻¹⁰</td>
<td>10⁻¹⁰</td>
<td>10⁻¹⁰</td>
<td>10⁻¹⁰</td>
<td>10⁻¹⁴</td>
<td>10⁻¹⁴</td>
</tr>
<tr>
<td>Initial friction angle</td>
<td>°</td>
<td>10.0</td>
<td>10.0</td>
<td>10.0</td>
<td>10.0</td>
<td>35.0</td>
<td>35.0</td>
</tr>
<tr>
<td>Final friction angle</td>
<td>°</td>
<td>35.0</td>
<td>35.0</td>
<td>35.0</td>
<td>40.0</td>
<td>35.0</td>
<td>35.0</td>
</tr>
<tr>
<td>$B_p$ coefficient</td>
<td></td>
<td>-</td>
<td>0.008</td>
<td>0.008</td>
<td>0.008</td>
<td>0.008</td>
<td>0.008</td>
</tr>
<tr>
<td>Cohesion</td>
<td>kPa</td>
<td>10.0</td>
<td>15.0</td>
<td>1.0</td>
<td>1.0</td>
<td>100.0</td>
<td>100.0</td>
</tr>
<tr>
<td>Dilatancy angle</td>
<td>°</td>
<td>0</td>
<td>0</td>
<td>3.2</td>
<td>3.2</td>
<td>2.0</td>
<td>2.0</td>
</tr>
</tbody>
</table>
### Petacciato soil parameters

<table>
<thead>
<tr>
<th>Soil parameters</th>
<th>Symbols</th>
<th>Unit</th>
<th>Blue-gray clay</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grain specific weight</td>
<td>( \rho_s )</td>
<td>kN/m(^3)</td>
<td>27.</td>
</tr>
<tr>
<td>Young modulus</td>
<td>( E )</td>
<td>MPa</td>
<td>95.0</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>( \nu )</td>
<td>-</td>
<td>0.21</td>
</tr>
<tr>
<td>Porosity</td>
<td>( n )</td>
<td>-</td>
<td>0.3</td>
</tr>
<tr>
<td>Intrinsic permeability</td>
<td>( k_w )</td>
<td>M(^2)</td>
<td>(10^{-17})</td>
</tr>
<tr>
<td>Initial friction angle</td>
<td>( \varphi_0 )</td>
<td>°</td>
<td>1.0</td>
</tr>
<tr>
<td>Final friction angle</td>
<td>( \varphi_f )</td>
<td>°</td>
<td>19.0</td>
</tr>
<tr>
<td>( B_p ) coefficient</td>
<td>( B_p )</td>
<td>-</td>
<td>0.01</td>
</tr>
<tr>
<td>Initial cohesion</td>
<td>( c_0 )</td>
<td>kPa</td>
<td>10</td>
</tr>
<tr>
<td>Final cohesion</td>
<td>( c_f )</td>
<td>kPa</td>
<td>171</td>
</tr>
<tr>
<td>( B_c ) coefficient</td>
<td>( B_c )</td>
<td>-</td>
<td>0.02</td>
</tr>
<tr>
<td>Dilatancy angle</td>
<td>( \psi_0 = \psi_f )</td>
<td>°</td>
<td>0</td>
</tr>
</tbody>
</table>
• Time dependent model of water transfer:
  – Pressure head: \( h_w = \frac{p_w}{\gamma_w} + y \)

  – Generalised Darcy’s law: \( \nu_w = -K_w(p_c)\nabla h_w \)

  – Richard’s equation: \( \frac{\partial \theta_w}{\partial t} = \nabla^T(K_w(p_c)\nabla h_w) \quad \text{(Darcy + water mass balance)} \)

With \( \theta_w = nS_{ew} \) (volume water content)

and \( S_{ew} = \left( \frac{V_w}{V_v} \right)_{\text{current}} \)
Hydro-mechanical coupling for the unsaturated soil

• Richards equation depends on 2 hydrodynamic characteristics:
  – Water retention curve of Van-Genuchten:
    \[
    S_{ew} = S_w + \frac{S_w - S_{rw}}{(1 + (\alpha p_c)^\beta)^{1-\beta}} \quad S_{rw} = \left(\frac{V_w}{V_v}\right)_{\text{minimal}} \quad S_w = \left(\frac{V_w}{V_v}\right)_{\text{maximal}}
    \]
  – Permeability: \( K_w = S_{ew} k_w \)

• Effective stress:
  \[\sigma' = \sigma - p_a \mathbb{1} + \kappa(p_a - p_w)\mathbb{1}\]

With: \( \kappa = S_{ew} \)
### Water retention curve for Trévoux

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Symbols</th>
<th>Unit</th>
<th>Sands</th>
<th>Marls</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximal degree of saturation</td>
<td>$S_w$</td>
<td>-</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Residual degree of saturation</td>
<td>$S_{rw}$</td>
<td>-</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>First retention parameter</td>
<td>$\alpha$</td>
<td>Pa$^{-1}$</td>
<td>$6.8 \times 10^{-5}$</td>
<td>$2.5 \times 10^{-5}$</td>
</tr>
<tr>
<td>Second retention parameter</td>
<td>$\beta$</td>
<td>-</td>
<td>4.8</td>
<td>2.0</td>
</tr>
</tbody>
</table>

![Water retention curve](image)

The graph illustrates the water retention curve for different soil types, showing the degree of saturation as a function of suction (kPa). The curves for sands and marls are clearly differentiated, indicating distinct soil behavior under varying suction conditions.
## Water retention curve for Petacciato

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Symbols</th>
<th>Unit</th>
<th>Clay</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximal degree of saturation</td>
<td>$S_{w}$</td>
<td>-</td>
<td>1.0</td>
</tr>
<tr>
<td>Residual degree of saturation</td>
<td>$S_{rw}$</td>
<td>-</td>
<td>0.1</td>
</tr>
<tr>
<td>First retention parameter</td>
<td>$\alpha$</td>
<td>Pa$^{-1}$</td>
<td>$1.0 \times 10^{-5}$</td>
</tr>
<tr>
<td>Second retention parameter</td>
<td>$\beta$</td>
<td>-</td>
<td>1.35</td>
</tr>
</tbody>
</table>

![Water retention curve graph](image-url)
• Expression of Hill’s stability criterion:
  – Local second order work: \( d^2W_{pi} = \sigma'_p i : \varepsilon_{pi} \)
  
  \[ d^2W_{norm.} = \frac{d^2W_{pi}}{\|\sigma'_p i\| \|\varepsilon_{pi}\|} \]

  – Global second order work: \( D^2W = \sum_{pi=1}^{N_{pi}} \sigma'_p i : \varepsilon_{pi} \omega_{pi} J_{pi} \)

  \[ D^2W_{norm.} = \frac{D^2W}{\sum_{pi=1}^{N_{pi}} \omega_{pi} J_{pi} \sum_{pi=1}^{N_{pi}} \|\sigma'_p i\| \|\varepsilon_{pi}\|} \]

  with:
  
  \( N_{pi} \): total number of integration points
  
  \( J_{pi} \): determinant of Jacobian transformation matrix for point \( pi \)
  
  \( \omega_{pi} \): weight factor for point \( pi \)
Application to landslides

• Loading program in the simulation:
  – Initial state: unsaturated soil (dry)
  – Progressive saturation by increasing the water table
  – Stability analysis thanks to second order work criterion

• Results:
  – Iso values of the second order work
  – Global second order work of the problem vs loading parameter
Application to Trévoux landslide

- Evolution of local second order work
  - Water rising modelling

\[ d^2W \text{ min} = -0.172 \]
\[ d^2W \text{ max} = 1.000 \]
Step 2
Application to Trévoux landslide

- Evolution of local second order work
  - Water rising modelling
Application to Trévoux landslide

- Evolution of global second order work: 

![Diagram](image)
Application to Petacciato landslide

- Evolution of local second order work
  - Water rising modelling
Application to Petacciato landslide

- Evolution of global second order work:
Locally, **Hill’s bifurcation criterion** comes before the other criteria (i.e. Mohr-Coulomb plastic limit condition, Rice’s localisation condition, etc.).

**Application of Hill’s criterion** to stability analyses of non-linear boundary problems:

- **Material scale**: local second order work criterion
  - detection of different failure modes (localized, diffuse),
  - description of the propagation of the potentially unstable zones.
- **Global scale**: global criterion by integrating of local second order work into considered volume
  - description of global stability of the whole body,
  - highlight of the influence of the parameters of the constitutive behaviour (saturation, hydraulic conductivity, etc.) and events of hydraulic nature (raining, water flow, earthquakes, etc.) or anthropic (constructions, excavations, etc.) to the global stability of body.