On non-coaxial stress-dilatancy theories

Sur les théories de non co-axialité contrainte/dilatance

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ABSTRACT: The influence of non-coaxiality between principal stresses and principal strain increments on mechanical behavior of soils has been investigated both experimentally and theoretically. In this paper, two non-coaxial stress-dilatancy theories for soils are considered. The theoretical frameworks are investigated and inconsistencies are pointed out. Then a possible way of reconciling these inconsistencies is proposed. Furthermore, a semi-empirical evolution equation is proposed for the degree of non-coaxiality.

RÉSUMÉ : L'influence, sur le comportement mécanique des sols, de la non co-axialité entre les contraintes principales et les déformations principales, est l'objet d'études tant expérimentales que théoriques. Dans le présent article, deux théories de non co-axialité contrainte/dilatance sont considérées. La structure théorique a été analysée, et certaines divergences ont pu être relevées entre les deux théories. Une solution pour les concilier est alors proposée. De plus, une équation semi-empirique est proposée pour exprimer le degré de non co-axialité.

KEYWORDS: non-coaxiality, non-coaxial plastic dissipation, stress-dilatancy

1 INTRODUCTION

Several stress-dilatancy formalisms assume coaxiality between principal stress and principal plastic strain increments. The two frequently applied stress-dilatancy formalisms are that follow from Taylor's (1948) work hypothesis and Rowe's (1962) stress-dilatancy theory. Both assume coaxiality. As shown in Biru and Benz (2012) the two approaches can be seen from a common point. In spite of this fact, the two approaches bear differences.

The possible influence of non-coaxiality on stress-dilatancy behavior of geomaterials has been first pointed out in de Jong (1976). Gutierrez and Ishihara (2000) introduced non-coaxiality into Taylor's work hypothesis. Later, Gutierrez and Wang (2009) introduced non-coaxiality to Rowe's stress-dilatancy theory. In this paper, the two non-coaxial approaches are investigated. Differences in the two approaches are pointed out and a possible way of reconciliation is proposed.

2 ON THE NON-COAXIAL TAYLOR AND ROWE STRESS DILATNCY RELATIONSHIPS

This section focuses on the non-coaxial Taylor (Gutierrez and Ishihara 2000) and non-coaxial Rowe (Gutierrez and Wang 2009) stress-dilatancy theories. The two approaches are investigated and differences are pointed out.

2.1 *Non-coaxiality for extended Taylor work hypothesis*

The non-coaxial version of extended Taylor work hypothesis (Gutierrez and Ishihara 2000), for triaxial compression, triaxial extension and 2D plane strain deformation modes, is given by

$$\tilde{\mathcal{D}}_M^p = p\dot{\varepsilon}_v^p + c_\Delta q\dot{\varepsilon}_q^p = M_{cv}^{m_s} p\dot{\varepsilon}_q^p , \tag{1}$$

where $\tilde{\mathcal{D}}_M^p$ is plastic dissipation and c_Δ is degree of (non-) coaxiality ,

$$p = \frac{r_1\sigma_1 + r_3\sigma_3}{r_1 + r_3} \;\&\; q = \frac{\sigma_1 - \sigma_3}{3 - r_1 r_3} \tag{2}$$

are mean stress and deviatoric stress respectively where σ_1 is the major principal stress and σ_3 is the minor principal stress, r_i are such that $r_1 = r_3 = 1$ for plane strain, $r_1 = 2r_3 = 2$ for triaxial extension and $2r_1 = r_3 = 2$ triaxial compression. The corresponding work conjugate strain rate measures,

$$\dot{\varepsilon}_v^p = r_1\dot{\varepsilon}_1^p + r_3\dot{\varepsilon}_3^p \;\&\; \dot{\varepsilon}_q^p = 2\frac{\dot{\varepsilon}_1^p - \dot{\varepsilon}_3^p}{r_1 + r_3} , \tag{3}$$

are volumetric strain rate and deviatoric strain rate, respectively; and

$$M_{cv}^{m_s} = \frac{3 r_1 r_3 \sin\varphi_{cv}}{3 - (r_3 - r_1)\sin\varphi_{cv}} , \tag{4}$$

where φ_{cv} is the friction angle at constant volumetric strain.

The stress-dilatancy relationship obtained by rearranging Eq. (1) is of the form

$$M_\psi = M_{cv}^{m_s} - c_\Delta M_\sigma , \tag{5}$$

where $M_\psi = \dot{\varepsilon}_v^p / \dot{\varepsilon}_q^p$ is dilatancy ratio and $M_\sigma = q/p$ is stress ratio.

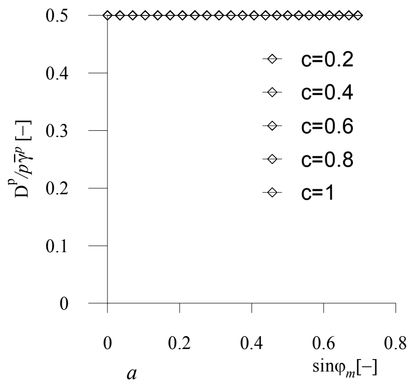
For plane strain condition, Eq. (5) simplifies to

$$\sin\hat{\psi}_m = \sin\varphi_c - c_\Delta \sin\varphi_m , \tag{6}$$

From this extension, the following points can be noted. Firstly, in this modification the plastic dissipation remains unaffected by non-coaxiality (See Figure 1*a*) but the stress-dilatancy relationship is. Secondly, the phase transformation line, *i.e.*, $\dot{\varepsilon}_v^p = 0$ requires

$$M_{pt} = \frac{1}{c_\Delta} M_{cv} \geq M_{cv} \,, \tag{7}$$

implying dependence on degree of non-coaxiality.



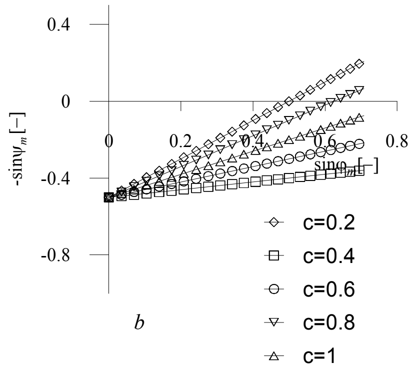


Figure 1: a) Normalized plastic dissipation rate according to the modified plastic dissipation in Eq. versus mobilized friction angle for different degrees of non-coaxiality $\varphi_c = 30^0$, b) Plots of dilatancy ratio versus stress ratio for different degrees of non-coaxiality, for Eq.

As can be seen from Figure 1*b*, the stress-dilatancy plots rotate around a value at zero stress ratio. Hence the higher the degree of non-coaxiality (*i.e.*, the less the value of c_Δ) the higher the phase transformation stress ratio is; consequently the more contractive the model behaves.

2.2 *Non-coaxiality for Rowe's stress-dilatancy theory*

Rowe (1962) assumed coaxiality between principal stresses and principal strain rates when he derived his stress-dilatancy relationship. De Jong (1976) questioned the validity of this assumption.

Gutierrez and Wang (2009), considering a plane strain condition, modified Rowe's (1962) stress-dilatancy theory for non-coaxiality which when reworked gives a non-coaxial plastic dissipation according to

$$\tilde{\mathcal{D}}_N^p = \sigma_1 \dot{\bar{\varepsilon}}_1^p + \sigma_3 \dot{\bar{\varepsilon}}_3^p = p c_\Delta \left(\sin\varphi_m + \sin\psi_{mR} \right) \dot{\gamma}^p \,, \tag{8}$$

where

$$\sin\psi_{mR} = -\frac{\sin\varphi_m - \sin\varphi_c}{1 - \sin\varphi_m \sin\varphi_c} \,, \tag{9}$$

σ_i are principal stress components and $\dot{\bar{\varepsilon}}_i^p$ are the conjugate plastic strain rate components projected along the principal stress components (coaxial components), $\dot{\gamma}^p = \dot{\varepsilon}_1^p - \dot{\varepsilon}_3^p$ is plastic shear strain rate and φ_m is mobilized friction angle; and φ_c is critical state friction angle. The resulting non-coaxial stress-dilatancy equation is

$$\sin\hat{\psi}_{mR} = c_\Delta \sin\psi_{mR} \,. \tag{10}$$

For $c_\Delta = 1$ and when the interparticle friction angle φ_μ in terms of the critical state friction angle φ_c, Rowe's original stress-dilatancy relationship is recovered.

From Eq. (8), the maximum plastic dissipation

$$\tilde{\mathcal{D}}_{N,\max}^p = 2 p c_\Delta \frac{1 - \cos\varphi_c}{\sin\varphi_c} \dot{\gamma}^p \,, \tag{11}$$

occurs when the mobilized friction ratio is

$$\sin\varphi_m = \sin\psi_{mR} = \frac{1 - \cos\varphi_c}{\sin\varphi_c} \,. \tag{12}$$

If Rowe's stress-dilatancy relationship, *i.e.*, Eq. (9) is approximated by

$$\sin\psi_{mR} \approx \sin\varphi_m - \sin\varphi_c \,. \tag{13}$$

such that the plastic dissipation in Eq. (8), simplifies to

$$\tilde{\mathcal{D}}_N^p \approx p c_\Delta \sin\varphi_c \dot{\gamma}^p \,. \tag{14}$$

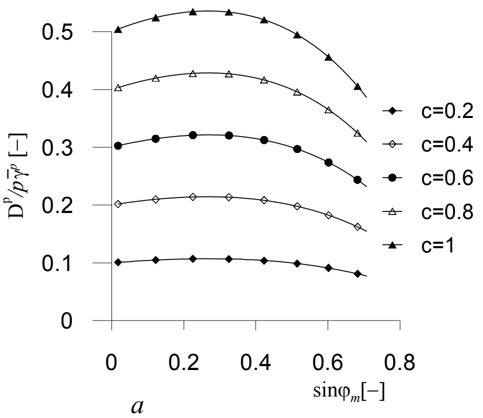
The non-coaxial dilatancy angle will then be

$$\sin\hat{\psi}_m \approx c_\Delta \left(\sin\varphi_m - \sin\varphi_c \right) . \tag{15}$$

Unlike the non-coaxial Taylor work hypothesis in Eq. (1) (Gutierrez and Ishihara 2000), the non-coaxial extension of Rowe's stress-dilatancy equation in Eq. (8) (Gutierrez and Wang 2009) or the simplified form given in Eq. (14) implies dependence of plastic dissipation on degree of non-coaxiality.

Furthermore, in the non-coaxial extension of Rowe's stress-dilatancy equation, the phase transformation remains unaffected by the degree of non-coaxiality, *i.e.*, for $\sin\hat{\psi}_{mR} = 0$ one obtains $\sin\psi_{mR} = 0 \rightarrow \sin\varphi_{pt} = \sin\varphi_c$.

For clarity, the normalized plastic dissipation rate and dilatancy ratio are plotted against the sine of mobilized friction angle, $\sin\varphi_m$, in Figure 2 for different values of degrees of non-coaxiality. Comparison of plots in Figure 1 and Figure 2 illustrates that the proposed theoretical modifications given in Eq. (1) and Eq. (8) fundamentally differ and their difference is too huge to ignore. This difference has been pointed out in Tsegaye *et al.* (2012).



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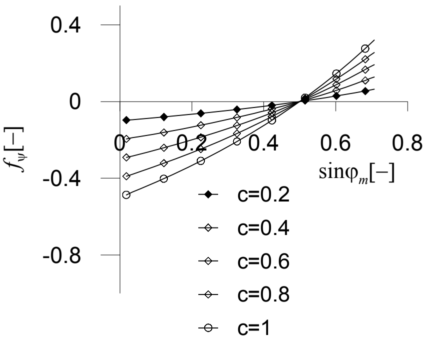

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Figure 2: a) Normalized plastic dissipation rate according to the modified plastic dissipation in Eq. versus mobilized friction angle for different degrees of non-coaxiality $\varphi_c = 30^0$; b) Plots of dilatancy ratio versus stress ratio for different degrees of non-coaxiality, for Eq.

2.3 *Proposal of a possible reconciliation*

The two approaches, as demonstrated for plane strain deformation mode, are not consistent with each other. The selection of one over the other may require experimental evidences. The authors do not come across sufficient experimental data to clearly choose one over the other. It is, however, self evident that both cannot hold to describe the same phenomenon. The authors propose an alternative work hypothesis,

$$\tilde{\mathcal{D}}_M^p = pc_\Delta M_{cv}^{m_s}\dot{\varepsilon}_q^p \, , \tag{16}$$

such that the difference is reconciled.

There are other ways by which Eq.(16) can be obtained, which shall not be discussed here. The resulting stress-dilatancy equation now becomes

$$M_\psi = c_\Delta\left(M_{cv}^{m_s} - M_\sigma\right) . \tag{17}$$

For plane strain deformation mode, Eq. (17) simplifies to

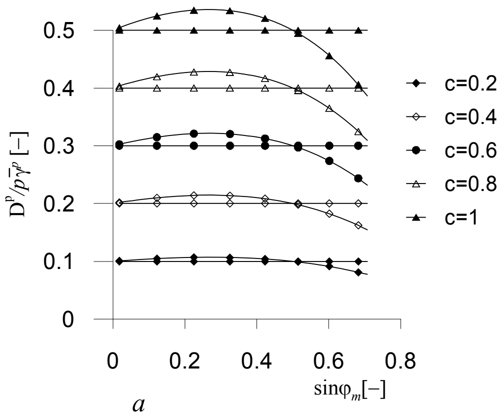
$$\sin\hat{\psi}_m = -c_\Delta\left(\sin\varphi_c - \sin\varphi_m\right) . \tag{18}$$

The hypotheses put forward by Gutierrez and Ishihara (2000) and Gutierrez and Wang (2009), amplify the differences between Taylor's work hypothesis and Rowe's minimum energy hypothesis. However, in the modification proposed here the difference between Eq.(10) and Eq. (18) is the same as that of the stress-dilatancy relationship from Taylor's work hypothesis and Rowe's minimum energy ratio hypothesis (Figure 3).

Note that although for the sake of simplicity a constant degree of non-coaxiality is used here; various experimental results demonstrate that the degree of non-coaxiality is an evolving state variable. Thornton and Zhang (2006) from their DEM simulations pointed out that "at any stage of shearing, during simple shear deformation, the angle of non-coaxiality depends on the mobilized angle of shearing resistance, the rate of dilation, the initial stress state, and the applied loading path". Post bifurcation evolution tendency of degree of non-coaxiality is controversial. For example, the tests by Vardoulakis and Georgopoulos (2004) show that degree of non-coaxiality vanishes even during post bifurcation deformation whereas Gutierrez and Vardoulakis (2007) show that degree of non-coaxiality increases during post bifurcation deformation

The authors find a semi-empirical equation for evolution of degree of non-coaxiality with stress ratio, for plane strain deformation mode, as

$$\delta c_\Delta = 2\pi^{R_\Delta} c_\Delta\left(1-c_\Delta\right)\frac{1}{C_N + N_\sigma}\delta N_\sigma \, , \tag{19}$$



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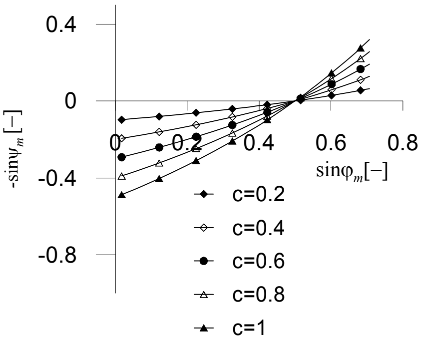

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Figure 3: a) Normalized plastic dissipation rate according to the modified plastic dissipation in Eq. versus mobilized friction angle for different degrees of non-coaxiality $\varphi_c = 30^0$; b) Plots of dilatancy ratio versus stress ratio for different degrees of non-coaxiality, for Eq.(18)

where

$$N_\sigma = \frac{\sigma_1}{\sigma_3} \text{ and } C_N = \tan^2\left(\frac{\pi}{4} + \frac{\varphi_c}{2}\right) \tag{20}$$

and R_Δ is a parameter that controls the rate at which the initial degree of non-coaxiality vanishes with stress ratio, Figure 4.

The equation implies that non-coaxiality vanishes with stress ratio and increases when $\delta N_\sigma < 0$. See for example Roscoe (1970), Arthur *et al.* (1986) for experimental justification.

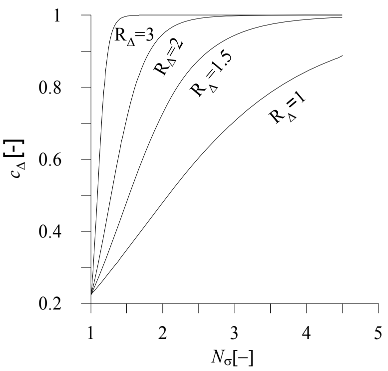

Figure 4: Evolution of degree of non-coaxiality with stress ratio, N_σ for $c_{\Delta 0} \approx 0.24$ and $C_{\tilde{N}} = 3$

3 CONCLUSION

In this paper, the inconsistencies between the non-coaxial extension of Taylor's work hypothesis (Gutierrez and Ishihara 2000) and Rowe's stress-dilatancy theory (Rowe 1962, Gutierrez and Wang 2009) are discussed. A new non-coaxial extended Taylor work hypothsis is proposed such that differences are reconsiled. A semi-emperical equation for evolution of non-coaxiality with stress ratio is proposed.

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