Semi-Analytical Solutions for Laterally Loaded Piles in Multilayered Soils

Solutions Semi-analytiques pour des pieux soumis à des charges latérales dans les sols multicouches

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ABSTRACT: Piles subjected to lateral forces and moments at the head are often analyzed in practice with the p-y method. However, the p-y method is not capable of capturing the complex three-dimensional interaction between the pile and the soil. The continuum approach is conceptually more appealing but it requires the use of numerical techniques, such as the three-dimensional (3D) finite element (FE) method. In order to save computational time, researchers have explored the development of closed-form solutions based on linear elasticity that can be used to obtain lateral pile deflection with depth. In this paper, we present semi-analytical methods developed to calculate the response of laterally loaded piles with general-shape cross sections embedded in multilayered elastic soil. The displacement field of the pile-soil system is taken to be the product of independent functions that vary in the vertical and horizontal directions. The differential equations governing the displacements of the pile-soil system are obtained using the principle of minimum total potential energy and calculus of variations. The input parameters needed for the analysis are the pile geometry, the soil profile, and the elastic constants of the soil and pile. The method produces results with accuracy comparable with that of a 3D FE analysis but requires much less computational time.

RÉSUMÉ : Les pieux soumis à des charges latérales et des moments à la tête sont souvent analysés dans la pratique par la méthode p-y. Toutefois, cette méthode n'est pas capable de prendre en compte les interactions complexes trois dimensionnelles entre le pieu et le sol. L'approche continue est conceptuellement plus attrayante, mais elle nécessite l'utilisation de techniques numériques, tels que les des éléments finis (EF) en trois dimensions (3D). Afin de gagner du temps de calcul, les chercheurs ont étudié le développement des solutions analytiques basées sur l'élasticité linéaire qui peuvent être utilisé pour évaluer les déplacements latérales du pieu avec la profondeur. Dans cet article, nous présentons des méthodes semi-analytiques développés pour calculer la réponse des pieux chargés latéralement avec des sections transversales en forme générale incorporés dans le sol élastique multicouche. Le champ de déplacement du système pieu-sol est considéré comme le produit de fonctions indépendantes qui varient dans les directions verticale et horizontale. Les équations différentielles qui régissent les déplacements du système pieu-sol sont obtenues en utilisant les principes variationnels et le principe de l'énergie potentielle totale minimum. Les paramètres d'entrée nécessaires pour l'analyse sont la géométrie du pieu, le profil du sol, et les constantes élastiques du sol et du pieu. Cette méthode donne des résultats d'une précision comparable à celle d'une analyse par éléments finis 3D mais nécessite beaucoup moins de temps de calcul.

KEYWORDS: laterally loaded single piles, elastic soil, continuum method, energy principles.

1 INTRODUCTION

Piles subjected to lateral forces and moments at the head are analyzed in practice with the p–y method (e.g., Reese and Cox, 1969). According to the p–y method, the pile is assumed to behave as an Euler–Bernoulli beam with the soil modeled as a series of discretely spaced springs, each connected to one of the pile segments into which the pile is discretized. The springs model the soil response to loading through p–y curves (p is the unit resistance per unit pile length offered by the springs, and y is the pile deflection), which are developed empirically by adjusting the curves until they match actual load–displacement results (e.g. Cox et al., 1974; Ashour & Norris, 2000). However, the p–y method often fails to predict pile response (Anderson et al., 2003; Kim et al., 2004), for it is not capable of capturing the complex three-dimensional interaction between the pile and the soil.

The continuum approach is conceptually more appealing; however, in order to model the soil as a continuum, the use of numerical techniques such as the three-dimensional (3D) finite element (FE) method, finite elements with Fourier analysis, the boundary element (BE) method or the finite difference (FD) method is often required (Poulos, 1971a, 1971b; Banerjee & Davis, 1978; Randolph, 1981). The 3D FE or FD methods can capture the most important features of the complex pile–soil interaction, but three-dimensional analyses are computationally expensive for routine practice.

In this paper, an analysis is developed, based on variational principles, by which the deflection, slope of the deflected curve, bending moment and shear force of laterally loaded piles with rectangular and circular cross section can be obtained. A multi-layered, elastic soil deposit is considered. The analysis captures the 3D pile–soil interaction and produces pile response using closed–form solutions. As a consequence of the analysis, the lateral response of piles can be obtained with a degree of rigor and speed not previously possible. The method can be extended to capture the non-linear pile response due to soil non-linearity.

2 PROBLEM DEFINITION

A circular cross-section pile with radius r_p and a rectangular cross-section pile with cross sectional dimensions $2a \times 2b$ and both with length L_p embedded in a multilayered soil profile are considered (Figure 1). Each soil layer extends to infinity in all horizontal directions, and the bottom layer extends to infinity in the downward direction. The pile is subjected to a horizontal force F_a and a moment M_a at the pile head. The goal of the analysis is to obtain pile deflection as a function of depth caused by the action of F_a and/or M_a at the pile head.

The soil medium is assumed to be an elastic, isotropic continuum, homogeneous within each layer, with Lame's constants G_s and λ_s. There is no slippage or separation between the pile and the surrounding soil or between the soil layers. The pile behaves as an Euler–Bernoulli beam with a constant flexural rigidity E_pI_p.

3 ANALYSIS

3.1 *Soil Displacement Field*

A separable variable technique is used to define the horizontal displacement fields in the soil, and the soil displacement u_z in the vertical direction is assumed to be negligible. The horizontal displacements for rectangular and circular piles are given by

$$\begin{cases} u_x = w(z)\phi_x(x)\phi_y(y), u_y = 0 & \text{for rectangular piles} \\ u_r = w(z)\phi_r(r)\cos\theta, u_\theta = -w(z)\phi_\theta(r)\sin\theta & \text{for circular piles} \end{cases} \tag{1}$$

where $w(z)$ is a displacement function (with a dimension of length) varying with depth z that describes the pile deflection, $\phi_x(x)$ and $\phi_y(y)$ are dimensionless displacement functions varying along the x and y directions of the Cartesian coordinate system used for the rectangular-cross section pile, and $\phi_r(r)$ and $\phi_\theta(r)$ are dimensionless displacement functions varying along the r and θ directions of the cylindrical coordinate system used for the circular-cross section pile (Figure 1). The dimensionless displacement functions describe how the displacements in the soil mass (due to pile deflection) decrease with increase in horizontal distance from the pile. These functions are set to unity at the pile-soil interface, which ensures perfect pile-soil contact, and are set to zero at infinite horizontal distance from the pile center, which ensures that the displacements in the soil due to the laterally loaded piles decrease as the horizontal distance from the pile increases.

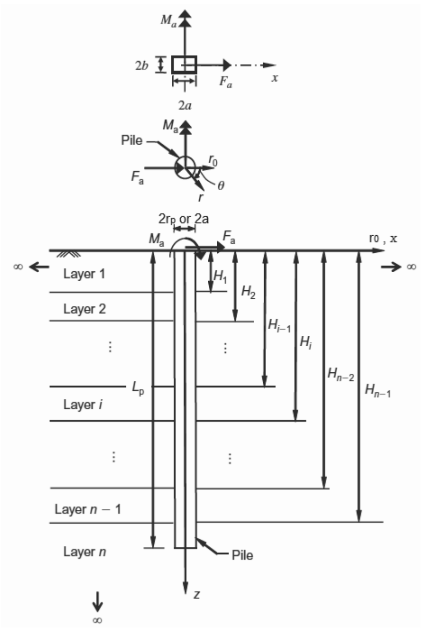


Figure 1. Laterally loaded pile in a layered elastic medium (Modified after Basu and Salgado 2008 and Basu et al. 2009)

3.2 *Potential Energy and its Minimization*

The total potential energy of the pile–soil system, including both the internal and external potential energies, is given by:

$$\Pi = \frac{1}{2}E_pI_p\int_0^{L_p}\left(\frac{d^2w}{dz^2}\right)^2 dz + \frac{1}{2}\int_{\Omega_s}\sigma_{mn}\varepsilon_{mn}d\Omega_s - F_a\, w\big|_{z=0} + M_a\frac{dw}{dz}\bigg|_{z=0} \tag{2}$$

where w is the lateral pile deflection; σ_{mn} and ε_{mn} are the stress and strain tensors in the soil, respectively (summation is implied by repetition of the indices m and n); and Ω_s represents the soil domain surrounding the pile that extends to infinity in the horizontal directions and from 0 to $+\infty$ in the z direction, but excludes the volume occupied by the pile. The first integral in Eq. (2) represents the internal potential energy of the pile, while the second integral represents the internal potential energy of the soil continuum. The remaining two terms represent the external potential energy.

The principle of minimum potential energy ($\delta\Pi = 0$) is used to obtain the differential equations governing the equilibrium condition of the pile-soil system:

$$\delta\Pi = E_pI_p\int_{z=0}^{z=L_p}\delta\left(\frac{d^2w(z)}{dz^2}\right)\left(\frac{d^2w(z)}{dz^2}\right)dz + \frac{1}{2}\int_{\Omega_s}\delta\left(\sigma_{mn}\varepsilon_{mn}\right)d\Omega_s - F_a\delta w(z)\big|_{z=0} + M_a\delta\left(\frac{dw(z)}{dz}\right)_{z=0} = 0 \tag{3}$$

The strain-displacement relationship for infinitesimal strains and the elastic stress-strain relationship, given by

$$\sigma_{mn} = 2G_s\varepsilon_{mn} + \lambda_s\varepsilon_{kk}\delta_{mn} \tag{4}$$

are used in Eq. (3) to express the first variation of the total pile-soil potential energy in terms of the soil elastic constants and the displacement functions described in Eq. (1). Thus Eq. (3) contains the first variations of the displacement functions w, ϕ_x and ϕ_y for rectangular pile and the first variations of the displacement functions w, $\phi_r(r)$ and $\phi_\theta(r)$ for circular pile. Since these variations are independent, the terms associated with each of these variations can be individually equated to zero, which produces the differential equations and boundary conditions of the displacement functions.

3.3 *Pile deflection*

The pile deflection equations corresponding to the i^{th} soil layer for both circular and rectangular piles is given by:

$$\begin{cases} E_pI_p\dfrac{d^4w_i}{dz^4} - t_i\dfrac{d^2w_i}{dz^2} + k_iw_i = 0 & 0 \le z \le L_p \\ -t_i\dfrac{d^2w_i}{dz^2} + k_iw_i = 0 & z \ge L_p \end{cases} \tag{5}$$

where, for rectangular piles:

$$t_i = \begin{cases} G_{s,i}\left(\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\phi_x^2\phi_y^2dxdy - 4ab\right) & 0 \le z \le L_p \\ G_{s,i}\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\phi_x^2\phi_y^2dxdy & z \ge L_p \end{cases} \tag{6}$$

$$k_i = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[(\lambda_{s,i} + 2G_{s,i})\left(\frac{d\phi_x}{dx}\right)^2\phi_y^2 + G_{s,i}\phi_x^2\left(\frac{d\phi_y}{dy}\right)^2\right]dxdy \tag{7}$$

and, for circular piles:

$$t_i = \begin{cases} \pi G_{s,i}\left[\int_{r_p}^{\infty}\left(\phi_r^2+\phi_\theta^2\right) r dr\right] & 0 \le z \le L_p \\ \pi G_{s,i}\left[\int_{r_p}^{\infty}\left(\phi_r^2+\phi_\theta^2\right) r dr + r_p^2\right] & z \ge L_p \end{cases} \quad (8)$$

$$\begin{aligned} k_i = \pi \Bigg[& (\lambda_{s,i}+2G_{s,i})\int_{r_p}^{\infty} r\left(\frac{d\phi_r}{dr}\right)^2 dr + G_{s,i}\int_{r_p}^{\infty} r\left(\frac{d\phi_\theta}{dr}\right)^2 dr \\ & +2\lambda_{s,i}\int_{r_p}^{\infty}(\phi_r-\phi_\theta)\frac{d\phi_r}{dr}dr + 2G_{s,i}\int_{r_p}^{\infty}(\phi_r-\phi_\theta)\frac{d\phi_\theta}{dr}dr \\ & +(\lambda_{s,i}+3G_{s,i})\int_{r_p}^{\infty}\frac{1}{r}(\phi_r-\phi_\theta)^2 dr \Bigg] \end{aligned} \quad (9)$$

The boundary conditions for the differential equations are given below:

At $z = 0$:

$$\begin{cases} w_1 = w_0 \quad \text{or} \quad E_p I_p \dfrac{d^3 w_1}{dz^3} - t_1 \dfrac{dw_1}{dz} = F_a \\ \text{and} \\ \dfrac{dw_1}{dz} = \theta_0 \quad \text{or} \quad E_p I_p \dfrac{d^2 w_1}{dz^2} = M_a \end{cases} \quad (10)$$

At $z = H_i < L_p$:

$$\begin{cases} w_i = w_{i+1} \\ E_p I_p \dfrac{d^3 w_i}{dz^3} - t_i \dfrac{dw_i}{dz} = E_p I_p \dfrac{d^3 w_{i+1}}{dz^3} - t_{i+1}\dfrac{dw_{i+1}}{dz} \\ \dfrac{dw_i}{dz} = \dfrac{dw_{i+1}}{dz} \\ E_p I_p \dfrac{d^2 w_i}{dz^2} = E_p I_p \dfrac{d^2 w_{i+1}}{dz^2} \end{cases} \quad (11)$$

At $z = L_p$, for free-base pile base:

$$\begin{cases} w_i = w_{i+1} \\ E_p I_p \dfrac{d^3 w_i}{dz^3} - t_i \dfrac{dw_i}{dz} = -t_{i+1}\dfrac{dw_{i+1}}{dz} \\ E_p I_p \dfrac{d^2 w_i}{dz^2} = 0 \end{cases} \quad (12)$$

At $z = L_p$, for fixed-base pile base:

$$\begin{cases} w_i = w_{i+1} = 0 \\ \dfrac{dw_i}{dz} = 0 \end{cases} \quad (13)$$

As $z \to \infty$,

$$w_i = 0 \quad (14)$$

The above differential equations are solved analytically after applying the boundary conditions to obtain the response of piles. The details of the solution can be found in Basu and Salgado (2008) and Basu et al. (2009).

3.4 *Soil displacement functions*

The differential equations for the soil displacement functions in the case of rectangular piles are given by:

$$\begin{cases} -p_x \dfrac{d^2\phi_x}{dx^2} + q_x \phi_x = 0 \\ -p_y \dfrac{d^2\phi_y}{dy^2} + q_y \phi_y = 0 \end{cases} \quad (15)$$

where,

$$p_x = \left[\sum_{i=1}^{n+1}(\lambda_{s,i}+2G_{s,i})\int_{H_{i-1}}^{H_i} w_i^2 dz\right]\int_{-\infty}^{+\infty}\phi_y^2 dy \quad (16)$$

$$q_x = \left[\sum_{i=1}^{n+1} G_{s,i}\int_{H_{i-1}}^{H_i} w_i^2 dz\right]\int_{-\infty}^{+\infty}\left(\frac{d\phi_y}{dy}\right)^2 dy + \left[\sum_{i=1}^{n+1} G_{s,i}\int_{H_{i-1}}^{H_i}\left(\frac{dw_i}{dz}\right)^2 dz\right]\int_{-\infty}^{+\infty}\phi_y^2 dy \quad (17)$$

$$p_y = \left[\sum_{i=1}^{n+1}(\lambda_{s,i}+2G_{s,i})\int_{H_{i-1}}^{H_i} w_i^2 dz\right]\int_{-\infty}^{+\infty}\phi_x^2 dx \quad (18)$$

$$q_y = \left[\sum_{i=1}^{n+1} G_{s,i}\int_{H_{i-1}}^{H_i} w_i^2 dz\right]\int_{-\infty}^{+\infty}\left(\frac{d\phi_x}{dx}\right)^2 dx + \left[\sum_{i=1}^{n+1} G_{s,i}\int_{H_{i-1}}^{H_i}\left(\frac{dw_i}{dz}\right)^2 dz\right]\int_{-\infty}^{+\infty}\phi_x^2 dx \quad (19)$$

The differential equations for the soil displacement functions in the case of circular piles are given by:

$$\begin{cases} \dfrac{d^2\phi_r}{dr^2} + \dfrac{1}{r}\dfrac{d\phi_r}{dr} - \left[\left(\dfrac{\gamma_1}{r}\right)^2 + \left(\dfrac{\gamma_2}{r_p}\right)^2\right]\phi_r = \dfrac{\gamma_3^2}{r}\dfrac{d\phi_\theta}{dr} - \left(\dfrac{\gamma_1}{r}\right)^2\phi_\theta \\ \dfrac{d^2\phi_\theta}{dr^2} + \dfrac{1}{r}\dfrac{d\phi_\theta}{dr} - \left[\left(\dfrac{\gamma_4}{r}\right)^2 + \left(\dfrac{\gamma_5}{r_p}\right)^2\right]\phi_\theta = -\dfrac{\gamma_6^2}{r}\dfrac{d\phi_\theta}{dr} - \left(\dfrac{\gamma_4}{r}\right)^2\phi_r \end{cases} \quad (20)$$

where $\gamma_1^2 = m_{s4}/m_{s1}, (\gamma_2/r_p)^2 = n_s/m_{s1}, \gamma_3^2 = (m_{s2}+m_{s3})/m_{s1}$

and $\gamma_4^2 = m_{s4}/m_{s2}, (\gamma_5/r_p)^2 = n_s/m_{s2}, \gamma_6^2 = (m_{s2}+m_{s3})/m_{s2}$

with

$$m_{s1} = \sum_{i=1}^{n+1}(\lambda_{s,i}+2G_{s,i})\int_{H_{i-1}}^{H_i} w_i^2 dz \quad (21)$$

$$m_{s2} = \sum_{i=1}^{n+1} G_{s,i}\int_{H_{i-1}}^{H_i} w_i^2 dz \quad (22)$$

$$m_{s3} = \sum_{i=1}^{n+1} \lambda_{s,i}\int_{H_{i-1}}^{H_i} w_i^2 dz \quad (23)$$

$$m_{s4} = \sum_{i=1}^{n+1}(\lambda_{s,i}+3G_{s,i})\int_{H_{i-1}}^{H_i} w_i^2 dz \quad (24)$$

$$n_s = \sum_{i=1}^{n+1} G_{s,i}\int_{H_{i-1}}^{H_i}\left(\frac{dw_i}{dz}\right)^2 dz \quad (25)$$

As mentioned earlier, these displacement functions are equal to unity at the pile-soil interface and they are equal to zero at the boundaries of the domain at infinity.

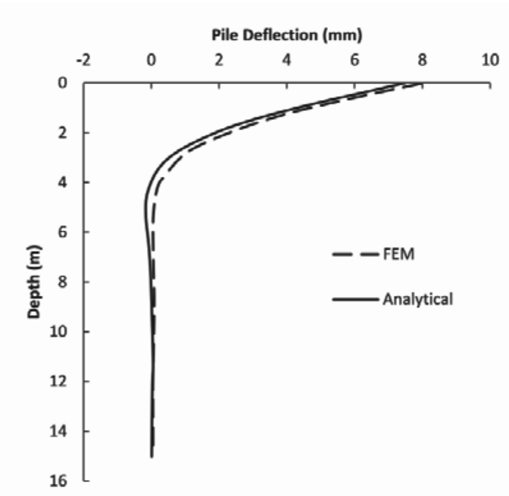
The above differential equations of the soil displacement functions for rectangular piles can be solved analytically as shown in Basu and Salgado (2008), while the coupled differential equations that govern the soil displacement surrounding the circular piles can be solved numerically using the finite difference method as shown in Basu et al. (2009).

As evident from Eqs. 5, 15, and 20, the responses of the pile and soil to the lateral loading are interrelated. Therefore, these differential equations are solved simultaneously following an iterative algorithm until they converge to a unique solution for a given soil profile, pile geometry and applied loading.

4 RESULTS

To illustrate the use of the analysis, a 15-m long drilled shaft with a diameter of 0.6 m and pile modulus E_p = 24 GPa, embedded in a four-layer soil deposit with H_1 = 2.0 m, H_2 = 5.0 m, and H_3 = 8.3 m; E_{s1} = 20 MPa, Es_2 = 35 MPa, E_{s3} = 50 MPa and E_{s4} = 80 MPa; ν_{s1} = 0.35, ν_{s2} = 0.25, ν_{s3} = 0.2 and ν_{s4} = 0.15 is considered (E_{si} and ν_{si} are the soil Young's modulus and Poisson's ratio for the i^{th} layer). A horizontal force F_a = 300 kN acts on the pile. The pile head and base are free to deflect and rotate. Figure 2 shows the pile deflection profile obtained using the present analysis and an analysis performed using the 3D FE method.

Figure 2. Deflection of a circular cross-section pile of 15 m length



As shown in Figure 2, the results match those of the FE analysis closely. The difference in the head deflection obtained from the present analysis and FE analysis is 6.6%.

Analyses were also performed on a square pile of 0.53m × 0.53m (which has the same flexural rigidity as that of the circular pile described above) embedded in the same soil profile as of Figure 2. Figure 3 compares the response of the square cross-section pile and the circular cross-section pile.

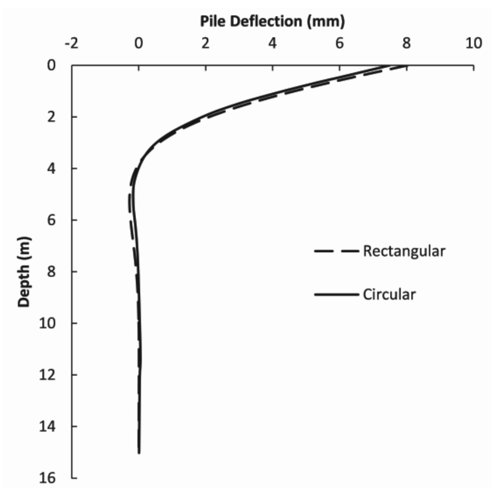


Figure 3. Deflection of a circular and rectangular cross-section piles of 15 m length and same flexural rigidity

Figure 3 shows that, if the second moment of inertia is the same for the piles, they will have (approximately) the same response under lateral loading even if the shapes of their cross sections are different. So, in summary, rectangular piles can be analyzed for lateral loads by replacing them with circular piles having the same second moment of inertia. However, this would work well for linear elastic soil in which knowledge of the appropriate soil constants is presumed, but would not be justified for an analysis that takes full account of soil non-linearity in which knowledge of the operative values of the soil 'constants' is not available a priori and must be obtained from the calculations themselves.

5 CONCLUSIONS

Analytical solutions for laterally loaded piles with rectangular and circular cross sections embedded in multilayered elastic media are obtained. The solutions produce the pile deflection, slope of the deflected curve, bending moment and shear force as functions of depth if the following are known: the pile cross-sectional dimensions and length, thicknesses of the soil layers, Young's modulus of the pile material, the Young's modulus and Poisson's ratio (or any pair of elastic constants) of the soils in the various layers, and the magnitudes of the applied force and moment. The governing differential equations for the pile deflections are obtained using the principle of minimum potential energy. The solution to all the governing differential equations is obtained iteratively and depends on the rate at which the displacements in the soil medium decreases with increasing distance from the pile. The shape of the pile cross section has a bearing on the pile response; however, it was shown that the piles with the same second moment of inertia produced the same response in elastic medium. The analysis presented in the paper can be used to make realistic predictions of the response of laterally loaded rectangular and circular piles.

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