A Variational Approach for Analysis of Piles Subjected to Torsion

Une approche variationnelle pour l'analyse des pieux soumis à torsion

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ABSTRACT: A framework is developed using the variational principles of mechanics for analyzing torsionally loaded piles in elastic soil. The total potential energy of the pile-soil system is minimized to obtain the differential equations governing the pile and soil displacements. Closed-form solutions are obtained for the angle of twist and torque in the pile as a function of depth. The analysis explicitly takes into account the three-dimensional pile-soil interaction in multi-layered soil. The results match well with the existing solutions and with those of equivalent finite element analyses.

RÉSUMÉ : Un cadre conceptuel est élaboré en utilisant les principles variationnels de la mécanique pour l'analyse de pieux chargés en torsion dans un sol élastique. L'énergie potentielle totale du système pieu-sol est minimisée pour obtenir les équations différentielles régissant le pieu et les déplacements du sol. Des solutions analytiques sont obtenues pour le couple et l'angle de torsion dans le pieu en fonction de la profondeur. L'analyse prend en compte explicitement les interactions pieu-sol tridimensionnelles dans un système multi-couches. Les résultats correspondent bien avec les solutions existantes ainsi qu' à celles obtenues par des analyses par éléments finis.

KEYWORDS: pile, torsion, multi-layered soil, elastic analysis, variational principles.

1 INTRODUCTION

Piles loaded laterally are often subjected to torsion due to eccentricities of applied lateral loads. The existing analysis methods are mostly based on numerical techniques such as the three-dimensional finite difference, finite element, discrete element or boundary element methods (Poulos 1975, Dutt and O'Neill 1983, Chow 1985, Basile 2010) although some analytical methods also exist mostly based on the subgrade-reaction approach (Randolph 1981, Hache and Valsangkar 1988, Rajapakse 1988, Budkowska and Szymcza 1993, Guo and Randolph 1996, Guo et al. 2007).

In this paper, a new analytical method is developed for torsionally loaded piles in multi-layered soil using the variational principles of mechanics. Based on a continuum approach, the analysis assumes a rational displacement field in the soil surrounding the pile, and explicitly captures the three-dimensional pile-soil interaction satisfying the compatibility and equilibrium between the pile and soil. Closed form solutions for the angle of twist and torque in the pile shaft are obtained. The analysis produces accurate results if the equivalent soil elastic modulus is correctly estimated.

2 ANALYSIS

A pile of radius r_p and length L_p embedded in a soil medium containing n layers is considered (Figure 1). The pile base rests in the n^{th} layer and the pile head is at the level of the ground surface. The pile has a shear modulus of G_p and is subjected to a torque T_a at the head. The soil layers extend to infinity in all horizontal directions and the bottom (n^{th}) layer extends to infinity in the downward vertical direction. The bottom of any layer i is at a depth of H_i from the ground surface; therefore, the thickness of the i^{th} layer is $H_i - H_{i-1}$ (note that $H_0 = 0$). The soil medium is assumed to be an elastic, isotropic continuum, homogeneous within each layer, characterized by Lame's constants λ_s and G_s. There is no slippage or separation between the pile and the surrounding soil or between the soil layers. For analysis, a polar (r-θ-z) coordinate system is assumed with its origin at the center of the pile head and z axis pointing downward.

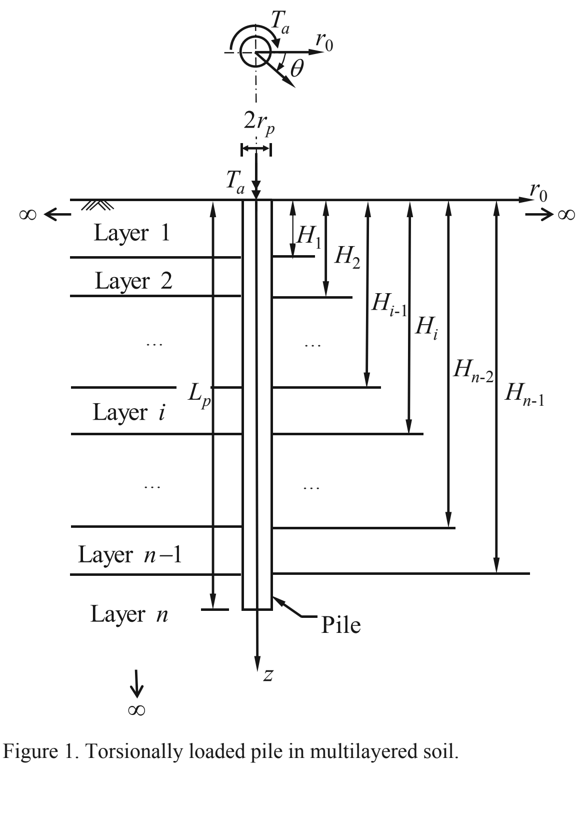


Figure 1. Torsionally loaded pile in multilayered soil.

Soil displacements u_r and u_z in the radial and vertical directions, generated by the applied torque T_a, can be assumed to be negligible (Figure 2). The tangential displacement u_θ in soil is nonzero and is assumed to be a product of separable variables as

$$u_\theta = w_p(z)\,\phi_s(r) = r_p\,\phi_p(z)\,\phi_s(r) \tag{1}$$

where w_p is the displacement in the tangential direction at the pile-soil interface (i.e., it is the tangential displacement at the outer surface of the pile shaft), ϕ_p is the angle of twist of the pile cross section ($w_p = r_p\phi_p$) which varies with depth z, and ϕ_s is a dimensionless function that describes how the soil displacement varies with radial distance r from the center of the pile. It is assumed that $\phi_s = 1$ at $r \le r_p$, which ensures no slip between pile and soil, and that $\phi_s = 0$ at $r = \infty$, which ensures that the soil displacement decreases with increasing radial distance from the pile.

Using the above soil displacement field, the strain-displacement and stress-strain relationships are used to obtain the total potential energy of the pile-soil system as

$$\Pi = \frac{1}{2}G_pJ_p\int_0^{L_p}\left(\frac{d\phi_p}{dz}\right)^2 dz + \frac{1}{2}\int_0^\infty\int_{r_p}^\infty\int_0^{2\pi}\left[G_s\left(r_p\frac{d\phi_p}{dr}\phi_s\right)^2 + G_s\left\{r_p\phi_p\left(\frac{d\phi_s}{dr}-\frac{\phi_s}{r}\right)\right\}^2\right]rd\theta drdz + \frac{1}{2}\int_{L_p}^\infty\int_0^{r_p}\int_0^{2\pi}\left[G_s\left(r_p\frac{d\phi_p}{dr}\right)^2 + G_s\left(\frac{r_p\phi_p}{r}\right)^2\right]rd\theta drdz - T_a\,\phi_p\big|_{z=0} \tag{2}$$

where J_p ($=\pi r_p^4/2$) is the polar moment of inertia of the pile cross section. Minimizing the potential energy (i.e., setting $\delta\Pi = 0$ where δ is the variational operator) produces the equilibrium equations for the pile-soil system. Using calculus of variations, the differential equations governing pile and soil displacements under equilibrium configuration are obtained.

The differential equation governing the angle of twist of pile cross section $\phi_p(z)$ within any layer i is obtained as

$$-\left(1+2\tilde{t}_i\right)\frac{d^2\phi_{pi}}{d\tilde{z}^2} + \tilde{k}_i\phi_{pi} = 0 \tag{3}$$

where

$$\tilde{t}_i = \begin{cases} \dfrac{\pi G_{si}r_p^2}{G_pJ_p}\displaystyle\int_{r_p}^\infty \phi_s^2 r dr & i = 1, 2, ..., n \\ \dfrac{\pi G_{sn}r_p^2}{G_pJ_p}\left[\dfrac{r_p^2}{2} + \displaystyle\int_{r_p}^\infty \phi_s^2 r dr\right] & i = n+1 \end{cases} \tag{4}$$

$$\tilde{k}_i = \begin{cases} \dfrac{2\pi G_{si}r_p^2L_p^2}{G_pJ_p}\displaystyle\int_{r_p}^\infty\left(\frac{d\phi_s}{dr}-\frac{\phi_s}{r}\right)^2 rdr & i = 1, 2, ..., n \\ \dfrac{2\pi G_{sn}r_p^2L_p^2}{G_pJ_p}\left[\ln r_p - \lim\limits_{\varepsilon\to 0}\left(\ln\varepsilon\right) + \displaystyle\int_{r_p}^\infty\left(\frac{d\phi_s}{dr}-\frac{\phi_s}{r}\right)^2 rdr\right] & i = n+1 \end{cases} \tag{5}$$

The boundary conditions of $\phi_p(z)$ are given by

$$-\left(1+2\tilde{t}_1\right)\frac{d\phi_{p1}}{d\tilde{z}} = \tilde{T}_a \tag{6}$$

at the pile head (i.e., at $z = \tilde{z} = 0$),

$$\phi_{pi} = \phi_{p(i+1)}$$

$$\left(1+2\tilde{t}_i\right)\frac{d\phi_{pi}}{d\tilde{z}} = \left(1+2\tilde{t}_{i+1}\right)\frac{d\phi_{p(i+1)}}{d\tilde{z}} \tag{7}$$

at the interface between any two layers (i.e., at $z = H_i$ or $\tilde{z} = \tilde{H}_i$), and

$$\left(1+2\tilde{t}_n\right)\frac{d\phi_{pn}}{d\tilde{z}} + \sqrt{2\tilde{k}_{n+1}\tilde{t}_{n+1}}\;\phi_{pn} = 0 \tag{8}$$

at the pile base (i.e., at $z = L_p$ or $\tilde{z} = 1$). The dimensionless terms in the above equations are defined as: $\tilde{T}_a = T_aL_p/G_pJ_p$; $\tilde{z} = z/L_p$ and $\tilde{H}_i = H_i/L_p$. In the above equations, the n^{th} (bottom) layer is split into two parts, with the part below the pile denoted by the subscript $n+1$; therefore, $H_n = L_p$ and $H_{n+1} = \infty$. In equation [5], $\tilde{k}_{n+1}$ is not defined at $r = 0$ as ln(0) is undefined; therefore, in obtaining the expression of $\tilde{k}_{n+1}$, the lower limit of integration was changed from $r = 0$ to $r = \varepsilon$ where ε is a small positive quantity (taken equal to 0.001 m in this study).

The general solution of equation (3) is given by

$$\phi_{pi}(\tilde{z}) = C_1^{(i)}\Phi_1 + C_2^{(i)}\Phi_2 \tag{9}$$

where $C_1^{(i)}$ and $C_2^{(i)}$ are integration constants of the i^{th} layer, and Φ_1 and Φ_2 are individual solutions of equation (3), given by

$$\Phi_1 = \sinh\beta_i\tilde{z} \tag{10a}$$

$$\Phi_2 = \cosh\beta_i\tilde{z} \tag{10b}$$

with

$$\beta_i = \sqrt{\frac{\tilde{k}_i}{1+2\tilde{t}_i}} \tag{11}$$

The constants $C_1^{(i)}$ and $C_2^{(i)}$ are determined for each layer using the boundary conditions given in equations (6)-(8).

The governing differential equation (3) resembles that of a column (or rod) supported by a torsional spring foundation undergoing a twist. The parameter $\tilde{t}_i$ accounts for the shear resistance of soil in the horizontal plane and $\tilde{k}_i$ represents the shear resistance of soil in the vertical plane. The torque $T(z)$ in the pile at any depth is given (in dimensionless form) by

$$\tilde{T}(\tilde{z}) = -\left(1+2\tilde{t}\right)\frac{d\phi_p}{d\tilde{z}} \tag{12}$$

where $\tilde{T}_a = T_aL_p/G_pJ_p$. The torque $T(z)$ includes the shear resistance offered by the horizontal planes of both the pile and surrounding soil. The governing differential equation (3) describes how the rate of change of this torque T with depth is balanced by the shear resistance in the vertical planes of the soil. The boundary conditions at the interfaces of the adjacent layers ensure continuity of angle of twist and equilibrium of torque across these horizontal planes. The boundary condition at the pile head ensures that equilibrium between the torque $T(z = 0)$ and applied torque T_a is satisfied. The boundary condition at the pile base ensures equilibrium by equating the torque in the pile and soil at a horizontal plane infinitesimally above the base with the torque in soil at a horizontal plane infinitesimally below the base.

The differential equation of $\phi_s(r)$ is given by

$$\frac{d^2\phi_s}{dr^2} + \frac{1}{r}\frac{d\phi_s}{dr} - \left[\frac{1}{r^2} + \left(\frac{\gamma}{r_p}\right)^2\right]\phi_s = 0 \tag{13}$$

where

(17a)

$$\gamma = \frac{r_p}{L_p}\sqrt{\frac{\sum_{i=1}^{n} G_{si}\int_{\tilde{H}_{i-1}}^{\tilde{H}_i}\left(\frac{d\phi_{pi}}{d\tilde{z}}\right)^2 d\tilde{z} + G_{sn}\left[\phi_{pn}^2\Big|_{\tilde{z}=1}\right]\sqrt{\frac{\tilde{k}_n}{8\tilde{t}_{n+1}}}}{\sum_{i=1}^{n} G_{si}\int_{\tilde{H}_{i-1}}^{\tilde{H}_i}\phi_{pi}^2 d\tilde{z} + G_{sn}\left[\phi_{pn}^2\Big|_{\tilde{z}=1}\right]\sqrt{\frac{\tilde{t}_{n+1}}{2\tilde{k}_n}}}} \tag{14}$$

At the boundaries $r = r_p$ and $r = \infty$, ϕ_s is prescribed as $\phi_s = 1$ and $\phi_s = 0$, respectively, which form the boundary conditions of equation (13).

The solution of equation (13) subjected to the above boundary conditions is given by

$$\phi_s = \frac{K_1\left(\frac{\gamma}{r_p} r\right)}{K_1(\gamma)} \tag{15}$$

where $K_1(\cdot)$ is the first-order modified Bessel function of the second kind. The dimensionless parameter γ determines the rate at which the displacement in the soil medium decreases with increasing radial distance from the pile.

Equations (3) and (15) were solved simultaneously following an iterative algorithm because the parameters involved in these equations are interdependent. At the same time, adjustments were made to the shear modulus by replacing G_s by an equivalent shear modulus $G_s^* = 0.5G_s$. This was necessary because the assumed soil displacement field described in equation (1) introduced artificial stiffness in the system and replacing G_s by G_s^* reduced this stiffness.

3 RESULTS

The accuracy of the proposed analysis is checked by comparing the results of the present analysis with those of previously obtained analyses and of three-dimensional (3D) finite element analyses performed as a part of this study. In order to compare the results with those of the existing solutions, normalized angle of twist at the pile head I_ϕ (also known as the torsional influence factor) and relative pile-soil stiffness π_t (Guo and Randolph 1996) are defined for piles in homogeneous soil deposits (with a constant shear modulus G_s)

$$I_\phi = \frac{\phi_p\big|_{z=0}}{\tilde{T}_a} = \phi_p\big|_{z=0}\frac{G_p J_p}{T_a L_p} \tag{16}$$

$$\pi_t = L_p\left(\frac{4\pi r_p^2 G_s}{G_p J_p}\right)^{\frac{1}{2}} \tag{17}$$

Figure 2 shows the plots of I_ϕ as a function of π_t for piles embedded in homogeneous soil, as obtained by Guo and Randolph (1996), Hache and Valsangkar (1978) and Poulos (1975) and as obtained from the present analysis. It is evident that the pile responses obtained from the present analysis match those obtained by others quite well. Figure 3 also shows that, for a given soil profile (in which G_s and G_p are constants) and a given applied torque T_a, I_ϕ of a slender pile is less than that of a stubby pile. Further, for a given pile geometry, I_ϕ increases as G_p/G_s increases.

In order to further check the accuracy of the present analysis, one example problem is solved and compared with the results of equivalent three-dimensional (3D) finite element analysis (performed using Abaqus). A four-layer deposit is considered in which a 30 m long pile with 1.0 m diameter is embedded. The top three layers are located over 0-5 m, 5-10 m and 10-20 m below the ground surface. The fourth layer extends down from 20 m to great depth. The elastic constants for the four layers are $G_{s1} = 8.6 \times 10^3$ kPa, $G_{s2} = 18.52 \times 10^3$ kPa, $G_{s3} = 28.8 \times 10^3$ kPa and $G_{s4} = 40 \times 10^3$ kPa, respectively. This results in $G_{s1}^* = 4.3 \times 10^3$ kPa, $G_{s2}^* = 9.26 \times 10^3$ kPa, $G_{s3}^* = 14.4 \times 10^3$ kPa and $G_{s4}^* = 20.0 \times 10^3$ kPa. The shear modulus of the pile $G_p = 9.6 \times 10^3$ kPa and the applied torque at the head $T_a = 100$ kN-m. Figure 3 shows the angle of twist in the piles as a function of depth for the two examples described above. It is evident that the match between the results of the present analysis and those of the finite element analyses is quite good.

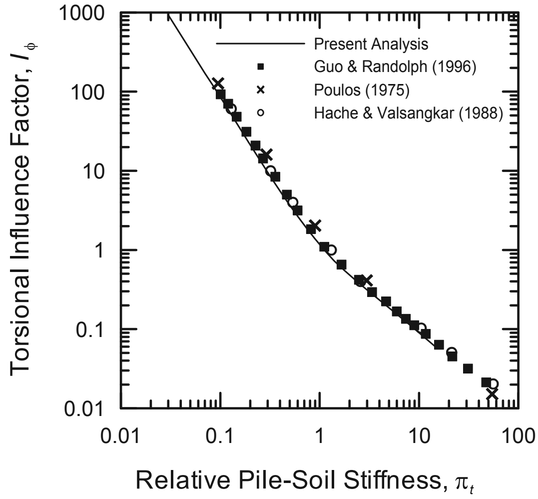


Figure 2. I_ϕ versus π_t for piles in homogeneous soil deposits.

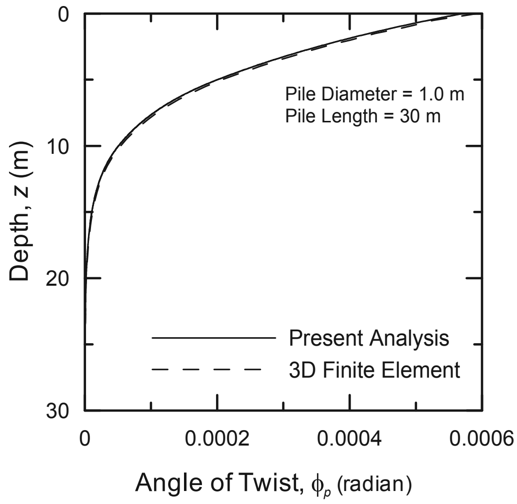


Figure 3. Angle of twist versus depth of a 10 m long pile in a 2-layer soil deposit.

The effect of soil layering is studied for piles in two-layer profiles with slenderness ratio $L_p/r_p = 20$ and 100 and for $G_p/G_{s1} = 1000$ (G_{s1} is the shear modulus of the top layer). I_ϕ is calculated using the above parameters for different values of H_1/L_p (H_1 is the thickness of the top layer) and G_{s2}/G_{s1} (G_{s2} is the shear modulus of the bottom layer). The values of I_ϕ thus obtained are normalized with respect to $I_{\phi,\text{homogeneous}}$ calculated for piles in homogeneous soil profiles with $G_s = G_{s1}$. Figure 4 shows the normalized parameter $I_\phi/I_{\phi,\text{homogeneous}}$ as a function of H_1/L_p. Note that $H_1/L_p = 0$ implies that the pile is embedded in a homogeneous soil with the shear modulus equal to G_{s2}. $H_1/L_p = 1$ implies that the entire pile shaft lies within the top layer and the pile base rests on top of the bottom layer. Also note that $I_{\phi,\text{homogeneous}}$ corresponds to the case where $H_1/L_p = \infty$. It is evident from Figure 4 that, for long, slender piles with $L_p/r_p = 100$, the presence of the second layer affects pile head response only if the bottom layer starts within the top 25% of the pile shaft. For short, stubby piles with $L_p/r_p = 20$, the head response is affected even if the bottom layer starts close to the pile base.

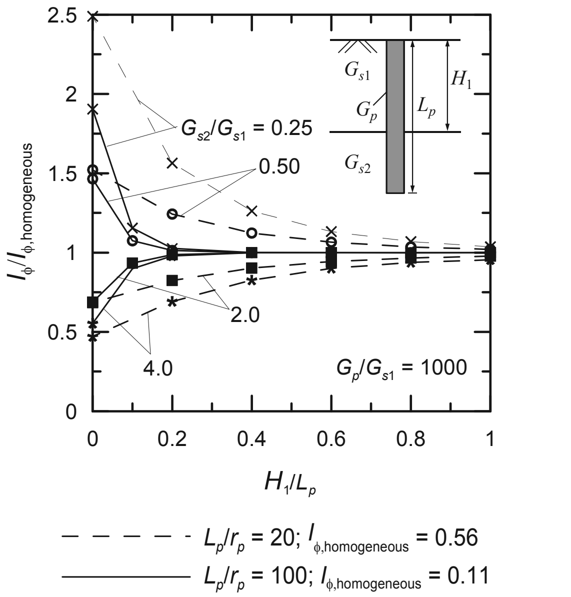


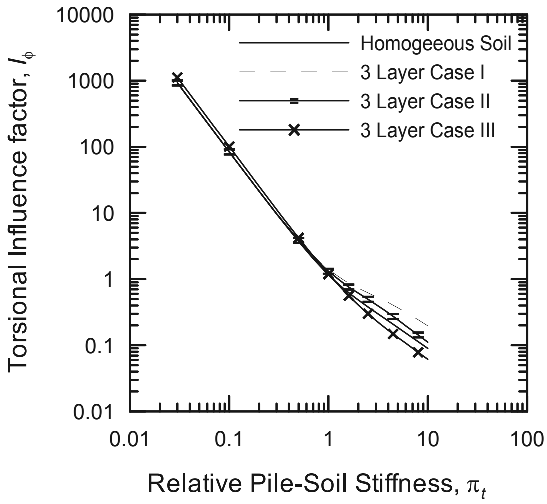
Figure 4. Angle of twist versus depth of a 10 m long pile in a 2-layer soil deposit.

The effect of soil layering is further studied with three-layer profiles for three different cases. For all the cases, the three layers divide the pile shaft into three equal parts of length $L_p/3$ and the pile base rests within the third layer, which extends down to great depth (Figure 5). Moreover, the shear moduli G_{s1}, G_{s2} and G_{s3} of the top, middle and bottom layers are so chosen that $(G_{s1} + G_{s2} + G_{s3})/3 = G_s$ for all the cases. Case I represents a soil profile in which the soil stiffness increases with depth — the top, middle and bottom (third) layer have a shear moduli equal to $0.23G_s$, $0.69G_s$ and $2.08G_s$, respectively. Note that, for this case, $G_{s3} = 3G_{s2}$ and $G_{s2} = 3G_{s1}$. For Case II, $G_{s1} = 0.69G_s$, $G_{s2} = 0.23G_s$ and $G_{s3} = 2.08G_s$. For case III, the soil stiffness decreases as depth increases with $G_{s1} = 2.08G_s$, $G_{s2} = 0.69G_s$ and $G_{s3} = 0.23G_s$. Figure 7 shows the I_ϕ versus π_t plots for these cases. The parameter π_t is calculated using the average shear modulus G_s. Also plotted in the figure is the I_ϕ versus π_t plots for homogeneous soil with shear modulus equal to G_s. As evident from Figure 5, the effect of layering is predominant for $\pi_t > 1.0$ for which $I_\phi < 1.0$. Similar plots can be obtained for cases with multiple layers.

4 CONCLUSIONS

The paper presents a method for analyzing piles in multi-layered elastic soil subject to a torque at the head. The analysis is based on a continuum approach in which a rational displacement field is defined and the variational principles of mechanics are used to develop the governing differential equations. The equations are solved analytically using which the pile response can be obtained using an iterative solution scheme.

The new method predicts the pile response quite accurately, as established by comparing the results of the present analysis with those obtained in previous studies by different researchers and with the results of equivalent three dimensional finite element analysis. A parametric study is performed for piles in layered soil profile. It is found that soil layering does have an effect on the pile response, particularly for short, stubby piles with low slenderness ratio.



G_p		Layering Cases I	II	III
G_{s1}	$L_p/3$	$0.23G_s$	$0.69G_s$	$2.08G_s$
G_{s2}	$L_p/3$	$0.69G_s$	$0.23G_s$	$0.69G_s$
G_{s3}	$L_p/3$	$2.08G_s$	$2.08G_s$	$0.23G_s$

Figure 5. Response of piles in three-layer soil.

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