Uniform effective stress equation for soil mechanics

Équation aux contraintes effectives uniformes pour la Mécanique des Sols

Shao L.-T., Liu G., Guo X.-X.
Dalian University of Technology, Dalian, P. R. China

ABSTRACT: The internal force of soil skeleton may, according to its different effect, be seperated into two groups of different balance system, including one by external load, and the pore fluid pressure and the skeleton internal force by it. In this paper we define the soil skeleton stress as the soil skeleton internal force by the external load excluding the pore fluid pressure. Then taking the soil skeleton, pore water and pore air as independent analysis object, we educe the soil differential equation of equilibrium from the balance analysis of infinitesimal. By comparing with the soil differential equation of equilibrium of total stress after adding up the soil differential equation of equilibrium of each phase of the soil, we can obtain the relationship expression of total soil stress, skeleton stress and pore fluid pressure, which is considered as the soil skeleton stress equation, equivalent to the effective stress equation by Terzaghi in the saturated condition. Therefore, the uniform soil mechanics effective stress equation is obtained, indicating the physical property of the effective stress equation is the interaction of inter-phase forces. The effective stress expression on the basis of the shearing strength equivalent or volume deformation equivalent can be expressed with the soil skeleton stress or pore fluid pressure as well.

RÉSUMÉ : Les efforts subis par le squelette solide d'un sol peuvent se diviser en deux parties en équilibres qui sont celles dues à le pression du fluide interstitiel et celle due au chargement externe. Dans cet article, dans l'analyse des efforts sur le squelette les effets de la pression interstitielle ne sont pas pris en compte dans un premier temps. Ensuite, on écrit les équations d'équilibre global sur un élément de volume infinitésimal en introduisant les effets des pressions de fluide et de gaz interstitiels. On analyse successivement l'équilibre de chaque phase et on additionne ensuite les équations ce qui permet de relier entre eux les efforts total, effective et de pression de fluide. L'équation des efforts du squelette représente, dans le cas saturé, l'équation sur les contraintes effectives de Karl Terzaghi. On trouve ainsi une équation unifiant des contraintes effectives en Mécanique des Sols. On l'utilise pour prouver que la nature physique de l'équation des contraintes effectives est de représenter l'interaction entre les différentes phases. On peut aussi établir de manière équivalente une expression de la contrainte effective basée sur la contrainte de cisaillement ou la déformation volumique.

KEYWORDS: Soil mechanics ; Effective stress equation ; Soil skeleton stress ; Saturated soil ; Unsaturated soil ; Equivalent stress

1 INTRODUCTION

The soil is the multiple-phase body of the soil skeleton and pore fluid, of which the former forms the soil structure The soil deformation and shearing strength is considered as the soil skeleton deformation and shearing strength.

The analysis shall, according to the present soil mechanics study method, be made on the soil internal force, with the whole soil including the pore fluid as the object and obtain the effective stress controlling the soil deformation and strength with the introduction of the effective stress principle. In terms of effective stress, some scholars consider it as the equivalent stress educed from the soil strength or deformation equivalent while some others regard it as the soil skeleton stress. However, the effective stress principle has no sufficient theoretical basis. Therefore, the principle has been in debate since it was put forward, with focus on the amendment and applicability of the effective stress equation for the saturated soil and whether there is the effective stress equation and its form for the unsaturated soil.

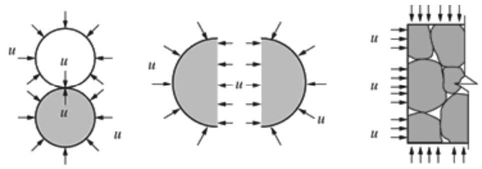
This thesis, taking the soil skeleton and pore water as independent analysis object, divided the force on the soil skeleton into two groups of balance system according to different effects. The thesis educes the internal force differential equation of equilibrium of the saturated and unsaturated soil with the interaction principle of inter-phase force, thus obtaining the soil skeleton stress equation, which indicates the soil skeleton stress equation is the soil effective stress equation. The thesis makes further discussion on the effective stress principle, indicating the effective stress equation by Terzaghi is unnecessary to be amended for it's accurately tenable for the saturated soil. The effective stress expression on the basis of the shearing strength equivalent or volume deformation equivalent may be expressed with the soil skeleton stress or pore fluid pressure as well. The study in the thesis may provide foundation for establishing the uniform soil mechanics theory of the saturated and unsaturated soil.

2 TWO BALANCE FORCE SYSTEMS ACTING ON SOIL SKELETON

To be definite, we consider the stress of the soil mixture as the total soil stress and the stress with the skeleton as the independent analysis object as the soil skeleton stress. In the internal force analysis on infinitesimal free body with the soil skeleton as the independent object, the acting forces shall be divided into two balance force systems, namely, the external load (excluding pore fluid pressure) and arising interaction forces between skeleton grains and the pore fluid pressure and arising interaction force between grains .

To place an isolated waterproof soil grain (as sand) in the water statically, if the pressure difference of different depth of the water is ignored, each point of the grain surface will bear the equivalent water pressure vertical to the surface. The (smoothed) internal force caused by the water pressure on any section of the grain is equivalent to the water pressure. Similarly, the average value of the stress of the soil skeleton grains resulting from the pore fluid pressure u_w on grain contact point (surface) shall be equivalent to u_w , as indicated in Figure 1(a). Therefore, whatever the shape and property of the soil grain contacting surface are, to investigate the effect of the fluid pressure, each soil skeleton grain may be considered to be an isolated grain in the fluid. Furthermore, the average stress caused by the pore fluid pressure on any section of the soil skeleton grain is equal to u_w , as indicated in Figure 1 (b). Thus, to take the soil skeleton as the free body, the average stress on the section caused by the pore fluid pressure is equal to the pore fluid pressure at the point, as indicated in Figure 1 (c).

In case of the pore fluid pressure, including pore water pressure or matrix suction acts on partial surface of the grain not on the whole surface of the grain, the skeleton grain still is in balance.



(a) (b) (c)

(a) stress on grain contact surface
(b) stress on the section of the soil grain
(c) stress on the section of the soil skeleton
Fig. 1 Stress of skeleton resulting from pore fluid pressure

The pore water pressure, pore air pressure and arising inter-grain action force of the grain system are in balance, without effect on the internal force of the internal force of the skeleton system. Meanwhile, the balance force system shall not affect the shearing strength and deformation of the skeleton system.

3 DIFFERENTIAL EQUATION OF EQUILIBRIUM AND SOIL SKELETON STRESS EQUATION OF SATURATED SOIL

The soil skeleton stress is defined to be the internal force resulting from the external forces excluding the pore fluid pressure acting on the soil skeleton of a unit area. Suppose the soil is homogeneous, to select the soil skeleton and pore water of the saturated soil as the free body of the independent analysis object for the internal force balance analysis with a group of inter-phase acting force, as indicated in Figure 2.

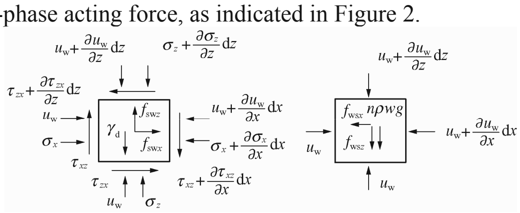


Fig. 2 Equilibrium analysis for solid and pore water phase

In Figure 2, n is the porosity of the soil, u_w the pore water pressure, $\sigma_x, \sigma_z, \tau_{xz}, \tau_{zx}$ positive stress and shearing stress respectively, $f_{\mathrm{sw}x}, f_{\mathrm{ws}x}, f_{\mathrm{sw}z}, f_{\mathrm{ws}z}$ the acting force and reacting force of the soil skeleton and pore water in the direction of x axis and z axis with same vale and opposite direction.

In the balance condition, the force acting on the skeleton and the pore water control its own state respectively. Therefore, the soil skeleton stress is also the effective stress to control the deformation and strength of the skeleton (or the soil body), which is the measurement of all external forces acting on the skeleton, exceeding the pore fluid pressure.

According to the internal force analysis figure, the equation of equilibrium of the soil skeleton and that of the pore water under the static balance state can by obtained respectively.

Soil skeleton:

$$\sigma_{ij,j} + (1-n)u_{\mathrm{w},i} - f_{\mathrm{sw}i} + X_{\mathrm{s}i} = 0 \tag{1}$$

Pore water:

$$nu_{\mathrm{w},i} + f_{\mathrm{sw}i} + X_{\mathrm{w}i} = 0 \tag{2}$$

Where, σ_{ij} is the soil skeleton stress, $i, j = x, y, z$, $X_{sx} = X_{sy} = 0$, $X_{sz} = \gamma_{\mathrm{d}}$, $X_{\mathrm{w}x} = X_{\mathrm{w}y} = 0$, $X_{\mathrm{w}z} = n\gamma_{\mathrm{w}}$.

To add formula (1) to (2), then obtain the equation of equilibrium after cancelling the terms of inter-phase acting force:

$$\sigma_{ij,j} + u_{\mathrm{w},i} + X_{\mathrm{sw}i} = 0 \tag{3}$$

where, $X_{\mathrm{sw}i} = X_{si} + X_{\mathrm{w}i}, X_{\mathrm{sw}x} = X_{\mathrm{sw}y} = 0, X_{\mathrm{sw}z} = r_{\mathrm{sat}}$.

Taking the soil skeleton and pore water as a whole system for the balance analysis, the differential equation of equilibrium of total soil stress in the static condition can be obtained:

$$\sigma_{\mathrm{t}ij,j} + X_{\mathrm{sw}i} = 0 \tag{4}$$

To compare formula (3) and (4), then

$$\sigma_{\mathrm{t}ij,j} = \sigma_{ij,j} + \delta_{ij}u_{\mathrm{w}} \tag{5}$$

where, $\sigma_{\mathrm{t}ij}$ is the total stress and δ_{ij} is Kronecker symbol.

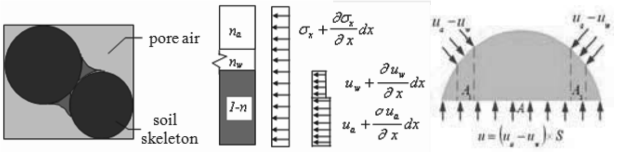
This is the saturated soil skeleton stress equation, consistent with the traditional effective stress equation, where the soil skeleton stress is the generally accepted soil effective stress.

The soil skeleton stress equation indicates the relationship between the total stress and the skeleton stress and pore water pressure, of which the physical property is the interaction of forces between the soil skeleton and pore water. From the deduction of the equation of equilibrium, it’s unnecessary to use the effective stress equation in the balance analysis with the soil skeleton and pore water as the free body separately. In other words, it’s required to introduce the soil skeleton stress equation to get the effective stress to control the soil skeleton deformation and strength in the force analysis on the whole structure of the soil skeleton and pore water to obtain the differential equation of equilibrium. Besides, it's noticeable that the equation (5) is applicable for saturated soil or porous materials with communicating pores filled with water, whatever the contacting property of grains is.

4 DIFFERENTIAL EQUATION OF EQUILIBRIUM AND SOIL SKELETON STRESS EQUATION OF UNSATURATED SOIL

The soil skeleton stress is still defined to be the internal force resulting from the external forces excluding the pore fluid pressure acting on the soil skeleton of a unit area for the unsaturated soil. Selecting free bodies for balance analysis requires meeting the following two conditions: □ the water and air in the communicating pores is immiscible;α the interacting force of the pore water and pore air is ignored. For simple and easy understanding, it may be supposed that the pore air pressure acts on the whole surface of the soil skeleton, just as on the saturated soil. The pressure difference (matrix suction) of pore water and pore air acts on the surface of the occupied by the pore water, as indicated in Figure 3(a).

Figure 3(b,c) indicates the force condition of the free body of unsaturated soil infinitesimal element and soil skeleton in the direction of x axis. For the homogeneous soil, the area ratios occupied by the pore water and pore air on the unit area are n_w / n and n_a / n respectively, n_w and n_a is the corresponding porosity of the phase of the pore water and that of the pore air.



(a) (b) (c)

(a) Unsaturated soil infinitesimal body
(b) Surface force of skeleton free body in direction of x axis
(c) Effect of matrix suction on soil skeleton
Fig. 3 Stress of unsaturated soil element and skeleton

Soil-water characteristic test shows that some content of water is always in the soil, however much the pressure (matrix suction) acting on the soil, which is considered as the residual water content, of which the corresponding saturation shall be S_r. The electro-mechanical interacting force of the pore water and the soil skeleton corresponding to the residual water content is so strong that the soil no longer shows the property of the fluid, but that of the solid or semi-solid. Therefore, in the force analysis on the unsaturated soil, the pore water corresponding to the residual water content may be considered as part of the soil skeleton. Now, the soil porosity is considered as n_e, porosity of pore water phase and pore air phase n_{ew} and n_{ea} respectively. The saturation without calculating the residual water content is the effective saturation, namely S_e, indicated with the formula:

$$S_e = \frac{S - S_r}{1 - S_r} \tag{6}$$

Similar to saturated soil, according to the internal force analysis figure, we can obtain the equation of equilibrium of each phase, and the equation of equilibrium of the soil element without any term of inter-phase acting force.

$$\sigma_{ij,j} + (S_e u_w)_{,i} + \left((1 - S_e)u_a\right)_{,i} + X_{sfi} = 0 \tag{13}$$

Or

$$\sigma_{ij,j} + u_{a,i} - \left(S(u_a - u_w)\right)_{,i} + X_{sfi} = 0 \tag{14}$$

To compare the equation (13) or (14) with the total stress equation of equilibrium, then obtain:

$$\sigma_{ij} = \sigma_{tij} - \delta_{ij} S_e u_w - \delta_{ij}(1 - S_e)u_a \tag{15}$$

Or

$$\sigma_{ij} = \sigma_{tij} - \delta_{ij} u_a + \delta_{ij} S_e (u_a - u_w) \tag{16}$$

This is the soil skeleton stress equation of unsaturated soil, or the relationship expression of the total soil stress, soil skeleton stress and pore water pressure and pore air pressure,

For saturated soil, $S_e = 1$, then the soil skeleton stress equation for unsaturated soil will be that of the saturated soil, or the effective stress equation by Terzaghi.

5 SOIL VOLUME CHANGING OR STRENGTH EQUIVALENT STRESS

Besides the soil skeleton stress, the forces on the soil skeleton also include the action of pore fluid pressure and arising internal force, which have different effect on the soil shearing strength and deformation. The latter only causes the volume deformation of the grains and pressure stress on contact points of soil grains, affecting the shearing strength of the soil. In case of fully considering the effect of the soil skeleton stress and pore fluid pressure, the shearing strength and volume changing expression for the unsaturated soil will be:

$$\tau_f = c' + \left[(\sigma_t - u_a) + S_e\left(u_a - u_w\right)\right]\tan\varphi' + a_c u_a \tan\psi - a_c S_e\left(u_a - u_w\right)\tan\psi \tag{17}$$

$$-\frac{\Delta V}{V} = C\cdot\left\{\Delta\left(\sigma_t - u_a\right) + \Delta\left[S_e\left(u_a - u_w\right)\right]\right\} + C_s\Delta u_a - C_s\Delta\left[S_e\left(u_a - u_w\right)\right] \tag{18}$$

which can be written into in the further:

$$\tau_f = c' + \left[\sigma_t - (1 - \frac{a_c \tan\psi}{\tan\varphi'})u_a + (1 - \frac{a_c\tan\psi}{\tan\varphi'})S_e\left(u_a - u_w\right)\right]\tan\varphi' \tag{19}$$

$$-\frac{\Delta V}{V} = C\cdot\left[\Delta\sigma_t - \left(1 - \frac{C_s}{C}\right)\Delta u_a + \left(1 - \frac{C_s}{C}\right)\Delta\left[S_e\left(u_a - u_w\right)\right]\right] \tag{20}$$

Where, φ and C_s is internal friction angle and coefficient of compressibility of the soil grains respectively; ϕ' and C the shearing strength and coefficient of compressibility of the soil. From it, obtain the equivalent stress expression for unsaturated soil:

(1) Shearing strength equivalent:

$$\sigma' = \sigma_t - (1 - \frac{a_c\tan\psi}{\tan\varphi'})\left[u_a - S_e\left(u_a - u_w\right)\right] \tag{21}$$

(2) Soil volume changing equivalent:

$$\Delta\sigma' = \Delta\sigma_t - \left(1 - \frac{C_s}{C}\right)\Delta\left[u_a - S_e\left(u_a - u_w\right)\right] \tag{22}$$

Generally, the pore air pressure in the soil is not high, the contacting area of soil grains is small and the value of α is close to zero. Then, the effect of the pore fluid pressure on the shearing strength and volume changing can be ignored, $\sigma' = \sigma_t - u_a + S_e\left(u_a - u_w\right)$, indicating only the effect of the soil skeleton stress.

When the pore air pressure $u_a = 0$, then

$$\begin{cases} \sigma' = \sigma_t - (1 - \frac{a_c\tan\psi}{\tan\varphi'})S_e u_w \\ \sigma' = \sigma_t - \left(1 - \frac{C_s}{C}\right)S_e u_w \end{cases} \tag{23}$$

When the soil is fully saturated, $S_e = 1$, the above mentioned formula (21) and (22) changes into the equivalent stress expression by Skempton.

According to the principle of causing equivalent volume changing on the soil infinitesimal or the shearing strength equivalent, A. W. Skempton educed the equivalent stress expression of the saturated soil and made experimental verification.

Shearing strength equivalent:

$$\sigma' = \sigma_t - (1 - \frac{a\cdot\tan\varphi}{\tan\phi'})u_w \tag{24}$$

Soil volume changing equivalent:

$$\sigma' = \sigma_t - (1 - \frac{C_s}{C})u_w \tag{25}$$

The equivalent stress expressions of the saturated soil and unsaturated soil are uniform.

6 EFFECT OF SOIL SKELETON STRESS EQUATION OF UNSATURATED SOIL

The above educed soil skeleton stress equation of unsaturated soil will be that of the saturated soil in the saturated condition, thus considered as the uniform soil skeleton stress equation of the saturated soil and unsaturated soil, which has two effects at least:

Firstly, to obtain the soil skeleton stress with the soil skeleton stress equation directly in the condition of learning the total stress and pore water and pore air pressure of any point in the soil;

Secondly, the skeleton stress is the soil effective stress.

The soil skeleton is the supporting phase of the soil or the structural phase of the soil, whose deformation and strength is the deformation and strength of the soil skeleton. The forces of the soil skeleton decide the strength and deformation of the soil skeleton. As above mentioned, the contribution of the pore water and pore air pressure on the soil strength and deformation can be ignored. Thus, the soil skeleton stress is the effective stress to decide the soil deformation and strength and the soil skeleton stress equation of the unsaturated soil can be considered as the effective stress equation of the unsaturated soil.

In terms of shearing strength of the unsaturated soil, Vanapalli and Fredlund gave the following shearing strength formula after the experiment and analysis:

$$\tau_f = c' + \left[\sigma_t - u_a + S_e\left(u_a - u_w\right)\right]\tan\phi' \tag{26}$$

We can find the stress expression in the square bracket of such formula is the above mentioned unsaturated soil skeleton stress, which indicates that the unsaturated soil shearing strength is controlled by the skeleton stress, as the same with that of saturated soil. The shearing strength formula of saturated soil and unsaturated soil is uniform with the concept of the soil skeleton stress.

No adequate experiment data is found on unsaturated soil volume changing.

7 DISCUSSION ON THE PRINCIPLE OF EFFECTIVE STRESS

The soil skeleton stress is the real internal force acting on the soil by the external load, or the effective stress by Terzaghi. The effective stress is not the virtual stress in such meaning. It is the real stress of the soil skeleton with definite physical meaning. The essence of the effective stress principle is that the soil skeleton stress decides the soil strength and deformation in case of ignoring the effect of the pore fluid pressure.

The effective stress principle is the most important one in the soil mechanic and the foundation of the modern soil mechanics.

Such principle, put forward by Terzaghi in 1936, states that the stress of any point at the soil section will be calculated with the total major stress σ_1 , σ_2 , σ_3 of such point. In case of the soil pores are filled with water under the stress of u , the total major stress consist of two parts: the first is u ,the stress acting on the water and solid with the equivalent strength in various directions, which is called the neutral stress or pore water pressure; the second is the difference of the total stress σ and the neutral stress, namely, $\sigma_1' = \sigma_1 - u$, $\sigma_2' = \sigma_2 - u$, $\sigma_3' = \sigma_3 - u$, which can act in the solid phase of the soil.

Such part of the total major stress is considered as the main effective stress. The changing neutral stress does not cause the volume changing actually. The neutral stress is not linked with the damaging soil in the stress conditions.

Porous materials (as sand, clay and concrete) is incompressible in the action to u , just like the internal friction equal to zero. The measured results of the changing stress such as compression deformation and changing shearing resistance are only caused by the changing effective stress σ_1' , σ_2' and σ_3' ."

To sum up, the principle of effective stress consists of two parts: the first is that the changing soil volume and shearing strength depends on the changing effective stress completely and the second is that the soil effective stress is equal to the difference of the total stress and pore water pressure.

The study in this thesis provides the theoretical foundation for the effective stress principle and also expresses that the effective stress principle by Terzaghi requires no further amendments and that the effective stress principle may be expanded to the unsaturated soil. Vanapalli and Fredlund made experiments and analysis, showing the soil skeleton stress (effective stress) controls the shearing strength of the unsaturated soil. No adequate experiment data is found on unsaturated soil volume changing.

8 CONCLUSIONS

(1) The effect of the pore water pressure and pore air pressure on the soil skeleton constitutes the balance force system respectively, keeping the soil skeleton in balance. If the soil skeleton stress is defined as the soil skeleton internal force from the external forces excluding the pore fluid pressure, the soil skeleton stress is the effective stress by Terzaghi. Therefore, the effective stress is not the virtual internal force but the real internal force strength of the soil skeleton.

(2) The effective stress equation by Terzaghi is tenable, unnecessary to make any amendment for the saturated soil, in case of ignoring the effect of the pore water pressure on the soil strength and volume changing.

(3) On the basis of the differential equation of equilibrium of the unsaturated soil, we can obtain the relationship expression of the skeleton stress (effective stress) from the external forces excluding pore water and pore air pressure and the total stress, pore water and pore air pressure, which is considered as the soil skeleton stress (effective stress) equation:

$$\sigma = \sigma_t - u_a + S_e\left(u_a - u_w\right)$$

The above formula will be the effective stress equation by Terzaghi for the saturated soil. The experiments and engineering experience show that the effective stress decide the strength and deformation of the saturated soil. The experiment results by Vanapalli et al prove that the effective stress decides the shearing strength of the unsaturated soil.

(4) If consideration is taken on the effect of the pore water and pore air pressure, the equivalent expression of the shearing strength and soil volume changing equivalent shall be respectively:

$$\sigma' = \sigma_t - (1 - \frac{a_c \tan\psi}{\tan\varphi'})\left[u_a - S_e\left(u_a - u_w\right)\right]$$

$$\Delta\sigma' = \Delta\sigma_t - \left(1 - \frac{C_s}{C}\right)\Delta\left[u_a - S_e\left(u_a - u_w\right)\right]$$

It will be the equivalent effective equation by Skempton in the saturated condition.

Therefore, the effective stress equation and equivalent stress equation of the unsaturated soil are the uniform effective stress equation and equivalent stress equation in the soil mechanics.

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